

# Existence of non-topological solutions for a skew-symmetric Chern-Simons system

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## Abstract

We investigate the existence of non-topological solutions  $(u_1, u_2)$  satisfying

$$u_i(x) = -2\beta_i \ln |x| + O(1), \quad \text{as } |x| \rightarrow +\infty,$$

such that  $\beta_i > 1$  and

$$(\beta_1 - 1)(\beta_2 - 1) > (N_1 + 1)(N_2 + 1),$$

for a skew-symmetric Chern-Simons system. By the bubbling analysis and the Leray-Schauder degree theory, we get the existence results except for a finite set of curves:

$$\frac{N_1}{\beta_1 + N_1} + \frac{N_2}{\beta_2 + N_2} = \frac{k-1}{k}, k = 2, \dots, \max(N_1, N_2).$$

This generalizes a previous work by Choe-Kim-Lin [9].

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## 1 Introduction

In this paper, we study the nonlinear elliptic system:

$$\begin{cases} \Delta u_1 + \lambda e^{u_2}(1 - e^{u_1}) = 4\pi \sum_{i=1}^{N_1} \delta_{p_{1i}}, \\ \Delta u_2 + \lambda e^{u_1}(1 - e^{u_2}) = 4\pi \sum_{i=1}^{N_2} \delta_{p_{2i}}, \end{cases} \quad \text{in } \mathbb{R}^2, \quad (1.1)$$

where  $\lambda > 0$  and  $\delta_p$  is the Dirac measure at  $p$ . System (1.1) arises in a relativistic Abelian Chern-Simons model involving two Higgs scalar fields and two gauge fields studied in [11, 15]. The Chern-Simons action density for this physics model is defined on the  $(2+1)$ -dimensional Minkowski space  $\mathbb{R}^{2,1}$  by

$$\mathcal{L} = -\frac{1}{4}\kappa\epsilon^{rst}A_r^{(1)}F_{st}^{(2)} - \frac{1}{4}\kappa\epsilon^{rst}A_r^{(2)}F_{st}^{(1)} + \overline{D_r\phi_i}D^r\phi_i - V(\phi_1, \phi_2),$$

where  $\phi_i, i = 1, 2$ , are two complex scalar field representing two Higgs particles of charges  $q_1$  and  $q_2$ .  $A_r^{(i)}, i = 1, 2$ , are two gauge fields with the induced electromagnetic fields  $F_{rs}^{(i)} = \partial_r A_s^{(i)} - \partial_s A_r^{(i)}$ ,

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$r, s = 0, 1, 2$ , and  $\kappa > 0$  is the coupling constant,  $D_r \phi^{(i)} = \partial_r \phi^{(i)} - \sqrt{-1} q_i A_r^{(i)} \phi^{(i)}$  is the covariant derivatives and

$$V(\phi_1, \phi_2) = \frac{q_1^2 q_2^2}{\kappa^2} [|\phi_2|^2(|\phi_1|^2 - c_1^2)^2 + |\phi_1|^2(|\phi_2|^2 - c_2^2)^2]$$

is the Higgs potential density. Even for a stationary solution, the Euler-Lagrangian equation for  $\mathcal{L}$  is very complicated. In [11, 15], the authors considered the minimizer for the associated energy, which satisfies the following self-dual equation:

$$\begin{aligned} D_1 \phi^{(i)} \pm D_2 \phi^{(i)} &= 0 \\ F_{12}^{(i)} \pm \frac{2q_i q_{i+1}^2}{\kappa^2} |\phi^{(i+1)}|^2 (|\phi^{(i)}|^2 - c_i^2) &= 0, \end{aligned} \quad (1.2)$$

where for the simplicity of notation, we let  $\phi^{(i)} \equiv \phi^{(j)}$ ,  $q_i \equiv q_j$  if  $i \equiv j \pmod{2}$ . Let  $u_i = \ln |\phi_i|^2$  and  $p_{ij}$  are the zeros of  $\phi_i$ ,  $i = 1, 2$ . Then system (1.2) can be transformed to (1.1).

We note that from the second equation of (1.2), both the quantities

$$\lambda \int_{\mathbb{R}^2} e^{u_2} (1 - e^{u_1}) dx \text{ and } \lambda \int_{\mathbb{R}^2} e^{u_1} (1 - e^{u_2}) dx \quad (1.3)$$

represent the total magnetic flux for both components  $u_i$ ,  $i = 1, 2$ . Therefore, from the physic's point of view, it is important for us to find solutions with finite integral of (1.3) for both components. In literature, a solution  $u = (u_1, u_2)$  is called topological if

$$u_i(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty, i = 1, 2,$$

and is called non-topological if

$$u_i(x) \rightarrow -\infty \quad \text{as } |x| \rightarrow +\infty, i = 1, 2.$$

If  $u$  is a topological solution, then  $u_i(x)$  decays exponentially to zero, thus, both  $e^{u_2}(1 - e^{u_1})$  and  $e^{u_1}(1 - e^{u_2}) \in L^1(\mathbb{R}^2)$ . If  $u$  is a non-topological solution, then one of  $e^{u_2}(1 - e^{u_1})$  and  $e^{u_1}(1 - e^{u_2})$  might be not in  $L^1(\mathbb{R}^2)$ . See [5].

For any configuration  $\{p_{11}, \dots, p_{1N_1}, p_{21}, \dots, p_{2N_2}\}$  in  $\mathbb{R}^2$ , the existence of a topological solution was obtained by Lin-Ponce-Yang [17]. The proof was rather complicated if we compare it with the case of  $SU(n)$  Chern-Simons system, as shown in [27], because system (1.1) is skew-symmetric, which is explained in the following.

To solve (1.1), we might first solve the following regularized form:

$$\begin{cases} \Delta u_1 + \lambda e^{u_2} (1 - e^{u_1}) = \sum_{i=1}^{N_1} \frac{4\kappa}{(\kappa + |x - p_{1i}|^2)^2}, \\ \Delta u_2 + \lambda e^{u_1} (1 - e^{u_2}) = \sum_{i=1}^{N_2} \frac{4\kappa}{(\kappa + |x - p_{2i}|^2)^2}, \end{cases} \quad (1.4)$$

where  $\kappa$  is a small positive constant. Then by letting  $\kappa \rightarrow 0$ , we could find a solution of (1.1). To apply the variational method, we introduce the background functions,

$$u_0^\kappa(x) = \sum_{i=1}^{N_1} \ln \left( \frac{\kappa + |x - p_{1i}|^2}{1 + |x - p_{1i}|^2} \right), v_0^\kappa(x) = \sum_{i=1}^{N_2} \ln \left( \frac{\kappa + |x - p_{2i}|^2}{1 + |x - p_{2i}|^2} \right).$$

Replacing  $u_i$  by  $u_1 - u_0^\kappa$  and  $u_2 - v_0^\kappa$ , the regularized form (1.4) becomes

$$\begin{cases} \Delta u_1 + \lambda e^{v_0^\kappa + u_2} (1 - e^{u_0^\kappa + u_1}) = h_1, \\ \Delta u_2 + \lambda e^{u_0^\kappa + u_1} (1 - e^{v_0^\kappa + u_2}) = h_2, \end{cases} \quad (1.5)$$

where  $h_1, h_2 \in W^{1,2}$  do not depend on  $\kappa > 0$ . By a direct computation, one can see that (1.5) is the Euler-Lagrange equation of the nonlinear functional:

$$I(u_1, u_2) = \int (\nabla u_1 \cdot \nabla u_2 + \lambda e^{u_0^\kappa + v_0^\kappa + u_1 + u_2} - \lambda e^{u_0^\kappa + u_1} - \lambda e^{v_0^\kappa + u_2} + h_2 u_1 + h_1 u_2) dx \quad (1.6)$$

From (1.6), the quadratic form  $\nabla u_1 \cdot \nabla u_2$  is not coercive, and this fact alone makes system (1.1) very difficult to study by variational method. Indeed, system (1.1) is a typical example of so called “skew-symmetric” system. For its precise definition and recent development, we refer [25, 26].

In this paper, we want to find the non-topological solutions with finite magnetic flux, i.e., for a given  $(\beta_1, \beta_2)$  with  $\beta_i > 1$ ,  $i = 1, 2$ , we want to find a solution  $(u_1, u_2)$  of (1.1) such that

$$u_i(x) = -2\beta_i \ln |x| + O(1), \quad i = 1, 2, \text{ near } \infty. \quad (1.7)$$

When  $u_1(x) = u_2(x)$ , system (1.1) is reduced to the Abelian Chern-Simons equation,

$$\Delta u + \lambda e^u (1 - e^u) = 4\pi \sum_{i=1}^N \delta_{p_i}. \quad (1.8)$$

Equation (1.8) has been extensively studied in the whole  $\mathbb{R}^2$  to search for topological solutions, non-topological solutions or in a flat torus to search for vortex condensates satisfying the periodic boundary condition, introduced by 't Hooft [23]. We refer reader to [3, 4, 7, 8, 9, 10, 13, 18, 20, 21, 22] and references therein for recent development. In particular, in [9], Choe-Kim-Lin proved the following existence theorem of non-topological solutions for (1.8).

**Theorem A.** [9] *Let  $p_1, \dots, p_N \in \mathbb{R}^2$  be given. For any number  $\beta > N + 2$  satisfying  $\beta \notin \{\frac{N(k+1)}{k-1} | k = 2, 3, \dots, N\}$ , there exists a solution  $u$  of (1.8) satisfying*

$$u(x) = -2\beta \ln |x| + O(1), \quad \text{near } \infty.$$

Our main result is to generalize Theorem A to system (1.1). First, we consider the case of (1.1) when all the vortex points collapse into one point and (1.1) becomes:

$$\begin{cases} \Delta u_1 + \lambda e^{u_2} (1 - e^{u_1}) = 4\pi N_1 \delta_0, \\ \Delta u_2 + \lambda e^{u_1} (1 - e^{u_2}) = 4\pi N_2 \delta_0, \end{cases} \quad \text{in } \mathbb{R}^2, \quad (1.9)$$

where  $N_1, N_2 \geq 0$  and  $\delta_0$  is the Dirac measure at 0. For (1.9), we consider  $u_i$  to be radially symmetric.

**Theorem 1.1.** *Given  $\beta_i > 1$ ,  $i = 1, 2$  satisfying*

$$(\beta_1 - 1)(\beta_2 - 1) > (N_1 + 1)(N_2 + 1), \quad (1.10)$$

*there exists a radial solution  $u = (u_1(r), u_2(r))$  of (1.9) such that (1.7) holds.*

Recently, Huang-Lin [12] considered the existence of non-topological solutions of (1.9) with  $N_1 = N_2 = 0$ . They showed that for any given pair  $(\beta_1, \beta_2)$  with  $1 < \beta_i < \infty$ ,  $i = 1, 2$  and  $(\beta_1 - 1)(\beta_2 - 1) > 1$ , there exists a unique non-topological radial solution of (1.9) with  $N_1 = N_2 = 0$  such that

$$u_i(x) = -2\beta_i \ln |x| + O(1), \quad i = 1, 2, \text{ near } \infty.$$

The proof of their result is based on non-degeneracy of linearized equations. Using this result, we may prove Theorem 1.1 by deforming system (1.9) by

$$\begin{cases} \Delta u_1 + \lambda e^{u_2} (1 - e^{u_1}) = 4\pi \epsilon N_1 \delta_0, \\ \Delta u_2 + \lambda e^{u_1} (1 - e^{u_2}) = 4\pi \epsilon N_2 \delta_0, \end{cases} \quad \text{in } \mathbb{R}^2 \quad (1.11)$$

for  $\epsilon \in [0, 1]$ . Suitably applying the Pohozaev's identity, we can show that for any solution  $(u_1, u_2)$  of (1.11) for fixed  $\beta_i$ ,  $\beta_i > 1$  such that (1.10) holds,  $u_i$  is uniformly bounded in some function space, which the classical Leray-Schauder degree (see [19]) can be applied. Therefore, the degree for (1.11) is invariant under the deformation. For  $\epsilon = 0$ , the degree is equal to  $-1$  due to the result of [12]. Thus, For (1.9) i.e.,  $\epsilon = 1$ , the degree is also equal to  $-1$ , and then the existence follows immediately. The complete proof will be given in Section 2.

For any configuration  $\{p_{11}, \dots, p_{1N_1}, p_{21}, \dots, p_{2N_2}\}$  in  $\mathbb{R}^2$ , we have the following theorem which extends Theorem A to system (1.1).

**Theorem 1.2.** *Let  $p_{11}, \dots, p_{1N_1}, p_{21}, \dots, p_{2N_2}$  be given. For any  $(\beta_1, \beta_2)$  satisfying (1.10) and*

$$\frac{N_1}{\beta_1 + N_1} + \frac{N_2}{\beta_2 + N_2} \notin \left\{ \frac{k-1}{k} \mid k = 2, \dots, \max(N_1, N_2) \right\}. \quad (1.12)$$

*Then there exists a solution  $(u_1, u_2)$  solves (1.1), (1.7).*

It is easy to see that if we take  $N_1 = N_2$ ,  $\beta_1 = \beta_2$  and  $\{p_{11}, \dots, p_{1N_1}\} = \{p_{21}, \dots, p_{2N_2}\}$ , then we can prove  $u_1 = u_2$ . In this case, Theorem 1.2 is the same as Theorem A.

Our proof uses the same strategy as Theorem 1.1, but the proof is more involved. The basic observation is that sometimes collapsing vortex points might not cause bubbling for system (1.1). Hence, as in the proof of Theorem 1.1, we want to establish a priori estimates of the following deformed system,

$$\begin{cases} \Delta u_1 + \lambda e^{u_2}(1 - e^{u_1}) = 4\pi \sum_{i=1}^{N_1} \delta_{\epsilon p_{1i}}, \\ \Delta u_2 + \lambda e^{u_1}(1 - e^{u_2}) = 4\pi \sum_{i=1}^{N_2} \delta_{\epsilon p_{2i}}, \end{cases} \quad \text{in } \mathbb{R}^2 \quad (1.13)$$

where  $\epsilon \in [0, 1]$ . For  $\epsilon = 1$ , it is just the same system as (1.1), and  $\epsilon = 0$ , (1.13) is reduced to (1.9). This a priori estimates could be obtained through the so-called ‘‘bubbling analysis’’. This kind of technique began with the celebrated work of Brezis-Merle [2] on the scalar nonlinear equation with exponential nonlinearity and has been developed into a powerful method through the works by Li-Shafrir [16], Bartolucci-Chen-Lin-Tarantello [1] and Chen-Lin [6]. For our situation, we have to extend [2] to the following system:

$$\begin{cases} \Delta u_{1n} + V_{1n} e^{u_{2n}} = 0, \\ \Delta u_{2n} + V_{2n} e^{u_{1n}} = 0, \end{cases} \quad \text{in } \Omega \subset \subset \mathbb{R}^2. \quad (1.14)$$

There is an interesting feature for bubbling solutions of (1.14): there exist a sequence of  $x_{1n} \rightarrow \bar{x}$  such that  $u_{1n}(x_{1n}) \rightarrow +\infty$  iff there exist a sequence of  $x_{2n} \rightarrow \bar{x}$  such that  $u_{2n}(x_{2n}) \rightarrow +\infty$ . Thus, one consequence of our analysis is that both  $V_{1n} e^{u_{2n}}$  and  $V_{2n} e^{u_{1n}}$  converges to  $\sum_{q \in S} M_q \delta_q$  and  $\sum_{q \in S} N_q \delta_q$  in measure, which implies the concentration phenomenon also occurs for system (1.1).

The paper is organized as follows. In Section 2, we will calculate the Leray-Schauder degree of the radially symmetric solutions for (1.11) and prove Theorem 1.1. We will generalize Brezis-Merle's alternative for PDE system (1.14) in Section 3. In Section 4, we will establish a Pohozaev's identity and then obtain the a priori estimates for (1.13) by bubbling analysis. Theorem 1.2 will be proved in Section 5 by Leray-Schauder degree theory.

## 2 Radially symmetric solutions

In this section, we will prove Theorem 1.1 by establishing a priori estimates of the radial solutions and using Leray-Schauder degree theory. We consider the following system:

$$\begin{cases} \Delta u_1 + e^{u_2}(1 - e^{u_1}) = 4\pi \epsilon N_1 \delta_0, \\ \Delta u_2 + e^{u_1}(1 - e^{u_2}) = 4\pi \epsilon N_2 \delta_0, \end{cases} \quad \text{in } \mathbb{R}^2 \quad (2.1)$$

for  $\epsilon \in [0, 1]$  where

$$u_i(x) = -2\beta_i \ln |x| + O(1), \quad \beta_i > 1, \quad i = 1, 2, \text{ near } \infty, \quad (2.2)$$

and

$$(\beta_1 - 1)(\beta_2 - 1) > (N_1 + 1)(N_2 + 1). \quad (2.3)$$

We begin our proof with the following Lemma.

**Lemma 2.1.** *Suppose  $(u_1(r), u_2(r))$  solves (2.1), (2.2), (2.3). Then*

$$|ru'_i(r)| \leq C, \quad \text{for } r > 0 \quad (2.4)$$

and

$$u_i(r) \leq -2 \ln r + C, \quad \text{for } r \geq 1 \quad (2.5)$$

where  $C$  is a constant depending only on  $\beta_i, N_i$ .

*Proof.* In fact, by directly integrating, we will have

$$ru'_1(r) - 2\epsilon N_1 = - \int_0^r e^{u_2} (1 - e^{u_1}) s ds, \quad ru'_2(r) - 2\epsilon N_2 = - \int_0^r e^{u_1} (1 - e^{u_2}) s ds.$$

(2.4) follows easily from these two equalities and Pohozaev's identity in Lemma 4.1. For  $s \geq r \geq 1$ , from (2.4), one gets

$$u_i(s) = u_i(r) + \int_r^s u'_i(\lambda) d\lambda \geq u_i(r) - C \ln \frac{s}{r}.$$

By Lemma 4.1, we have

$$C' \geq \int_r^\infty s e^{u_i(s)} ds \geq \int_r^\infty s e^{u_i(r)} \left(\frac{r}{s}\right)^C ds = r^2 e^{u_i(r)} \int_1^\infty t^{1-C} dt \geq cr^2 e^{u_i(r)}.$$

This proves (2.5). □

**Lemma 2.2.** *Suppose  $(u_1(r), u_2(r))$  solves (2.1), (2.2), (2.3). Then*

$$|u_1|_{L^\infty(K)} + |u_2|_{L^\infty(K)} \leq C_K, \quad \forall K \subset \subset \mathbb{R}^2 \setminus \{0\}.$$

*Proof.* If not, one may assume that there exists a sequence of positive numbers  $r_n \rightarrow r^* > 0$  and  $\epsilon_n \rightarrow \epsilon^*$  such that  $u_{1n}(r^*) \rightarrow -\infty$  as  $n \rightarrow \infty$ . From (2.5), one gets  $u_{1n}(r) \rightarrow -\infty$  as  $r \rightarrow +\infty$  uniformly for all  $n$ . Hence, we have  $\max_{\mathbb{R}^2} u_{1n} \rightarrow -\infty$  as  $n \rightarrow \infty$ . By Pohozaev's identity in Lemma 4.1, we have

$$o(1) = \max_{\mathbb{R}^2} e^{u_{1n}} \int_{\mathbb{R}^2} e^{u_{2n}} dx \geq \int_{\mathbb{R}^2} e^{u_{1n} + u_{2n}} dx = 4\pi((\beta_1 - 1)(\beta_2 - 1) - (\epsilon_n N_1 + 1)(\epsilon_n N_2 + 1)) \geq c_0 > 0.$$

This yields a contradiction. □

Set  $h_{i\epsilon} = 2\epsilon N_i \ln r - (\beta_i + \epsilon N_i) \ln(1 + r^2)$ ,  $i = 1, 2$ .

**Lemma 2.3.** *Suppose  $(u_1(r), u_2(r))$  solves (2.1), (2.2), (2.3). Then*

$$|u_1 - h_{1\epsilon}|_{L^\infty(\mathbb{R}^2)} + |u_2 - h_{2\epsilon}|_{L^\infty(\mathbb{R}^2)} \leq C.$$

*Proof.* Let  $v_1, v_2$  be defined as  $v_i = u_i - h_{i\epsilon}$ ,  $i = 1, 2$ . By a direct computation,  $v_i$  should satisfy

$$\begin{cases} \Delta v_1 + e^{u_2}(1 - e^{u_1}) = \frac{4(\epsilon N_1 + \beta_1)}{(1 + |x|^2)^2}, \\ \Delta v_2 + e^{u_1}(1 - e^{u_2}) = \frac{4(\epsilon N_2 + \beta_2)}{(1 + |x|^2)^2}, \end{cases} \quad \text{in } \mathbb{R}^2. \quad (2.6)$$

Since  $u_1, u_2 < 0$  and  $u_1, u_2$  are uniformly bounded in  $L^\infty(\partial B_1)$ , we can take  $\eta(x) = C|x|^2$  as a barrier function for some suitable large constant  $C$ . This proves that  $v_i \in L^\infty(B_1)$ ,  $i = 1, 2$ . In order to show  $v_i \in L^\infty(|x| \geq 1)$ , one should take Kelvin transformation as follows:

$$\xi_i(x) = u_i(x/|x|^2) - 2\beta_i \ln |x|, \quad |x| \leq 1, \quad i = 1, 2.$$

Then  $\xi_i$  satisfies

$$\begin{aligned} \Delta \xi_1 + |x|^{2\beta_2-4} e^{\xi_2} (1 - |x|^{2\beta_1} e^{\xi_1}) &= 0, \\ \Delta \xi_2 + |x|^{2\beta_1-4} e^{\xi_1} (1 - |x|^{2\beta_2} e^{\xi_2}) &= 0, \end{aligned} \quad \text{in } |x| \leq 1.$$

We need to show  $\xi_i$  is bounded from above. If not, we may assume  $\max_{B_1} \xi_{1n} = \xi_{1n}(r_n) \rightarrow +\infty$ ,  $r_n \rightarrow 0$ ,  $\epsilon_n \rightarrow \epsilon^*$  as  $n \rightarrow \infty$ . Set

$$s_n = \min\{e^{-\frac{\lambda_{1n}}{2\beta_1-2}}, e^{-\frac{\lambda_{2n}}{2\beta_2-2}}\} = e^{-\frac{\lambda_{1n}}{2\beta_1-2}} \rightarrow 0,$$

where  $\lambda_{in} = \max_{B_1} \xi_{in}$ ,  $i = 1, 2$ . In fact, By (2.5), we will have  $\xi_{1n}(r_n) + 2(\beta_1 - 1) \ln r_n \leq C$  which means  $\frac{r_n}{s_n} \leq C$ .

Set

$$\bar{\xi}_{in}(x) = \xi_{in}(s_n x) - \lambda_{in}, \quad i = 1, 2.$$

Then we have  $\bar{\xi}_{in} \leq 0$ ,  $i = 1, 2$ , and

$$\begin{cases} -\Delta \bar{\xi}_{1n} = e^{\lambda_{2n} + 2(\beta_2 - 1) \ln s_n} |x|^{2\beta_2-4} e^{\bar{\xi}_{2n}} (1 - r_n^2 s_n^2 |x|^{2\beta_1} e^{\bar{\xi}_{1n}}), \\ -\Delta \bar{\xi}_{2n} = |x|^{2\beta_1-4} e^{\bar{\xi}_{1n}} (1 - r_n^2 s_n^2 e^{\lambda_{2n}} |x|^{2\beta_2} e^{\bar{\xi}_{2n}}), \end{cases} \quad \text{in } |x| \leq 1/s_n.$$

Also we have  $\bar{\xi}_{1n}(x) \leq \bar{\xi}_{1n}(\frac{r_n}{s_n}) = 0$ . Then by standard elliptic estimates and  $\lambda_{2n} + 2(\beta_2 - 1) \ln s_n \leq 0$ , we have  $\bar{\xi}_{1n} \rightarrow \bar{\xi}_1$  in  $W_{loc}^{2,\gamma}(\mathbb{R}^2)$  for some  $\gamma > 1$ . As for  $\bar{\xi}_{2n}$ , by Harnack inequality, one gets either  $\bar{\xi}_{2n} \rightarrow -\infty$  locally uniformly in  $\mathbb{R}^2$  or  $\bar{\xi}_{2n} \rightarrow \bar{\xi}_2$  in  $W_{loc}^{2,\gamma}(\mathbb{R}^2)$ . By  $\int_{\mathbb{R}^2} |x|^{2\beta_1-4} e^{\bar{\xi}_1} dx < +\infty$ , we can exclude the previous case. In fact, by the same arguments, we also have  $\lambda_{2n} + 2(\beta_2 - 1) \ln s_n \geq -C$ , otherwise,

$$\Delta \bar{\xi}_1 = 0, \bar{\xi}_1(x_0) = 0, x_0 = \lim_{n \rightarrow \infty} \frac{r_n}{s_n} \Rightarrow \bar{\xi}_1 \equiv 0$$

which contradicts to  $\int_{\mathbb{R}^2} |x|^{2\beta_1-4} e^{\bar{\xi}_1} dx < +\infty$  by Fatou's lemma. And  $\bar{\xi}_1, \bar{\xi}_2$  satisfy

$$\begin{cases} -\Delta \bar{\xi}_1 = c_0 |x|^{2\beta_2-4} e^{\bar{\xi}_2}, \\ -\Delta \bar{\xi}_2 = |x|^{2\beta_1-4} e^{\bar{\xi}_1}, \end{cases} \quad \text{in } \mathbb{R}^2$$

with  $|x|^{2\beta_1-4} e^{\bar{\xi}_1}, |x|^{2\beta_2-4} e^{\bar{\xi}_2} \in L^1(\mathbb{R}^2)$  for some constant  $0 < c_0 \leq 1$ . Set

$$A_1 = \frac{1}{2\pi} \int_{\mathbb{R}^2} c_0 |x|^{2\beta_2-4} e^{\bar{\xi}_2} dx, A_2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2\beta_1-4} e^{\bar{\xi}_1} dx.$$

By Pohozaev's identity and repeating the arguments in Lemma 4.1, one gets

$$A_1 A_2 = 2(\beta_2 - 1) A_1 + 2(\beta_1 - 1) A_2.$$

Noting  $2(\beta_1 + \epsilon^* N_1) \geq A_1, 2(\beta_2 + \epsilon^* N_2) \geq A_2$ , we have

$$\frac{\beta_1 - 1}{\beta_1 + \epsilon^* N_1} + \frac{\beta_2 - 1}{\beta_2 + \epsilon^* N_2} \leq 1, \quad \text{i.e., } \beta_1 \beta_2 - \beta_1 - \beta_2 \leq \epsilon^* (\epsilon^* N_1 N_2 + N_1 + N_2)$$

which contradicts to (2.3). This proves the upper bound of  $\xi_i$ . By Harnack inequality and  $\xi_i \in L^\infty(\partial B_1)$ , we get  $\xi_i$  is bounded in  $L^\infty(B_1)$ . This proves Lemma 2.3.  $\square$

By Lemma 2.3, we now can calculate the corresponding Leray-Schauder degree of (2.6). Introduce the following Hilbert space for  $\beta = \min(\beta_1, \beta_2, 2) > 1$

$$\mathcal{D} = \{v : \mathbb{R}^2 \rightarrow \mathbb{R} \mid |v|_{\mathcal{D}}^2 = \int_{\mathbb{R}^2} |\nabla v|^2 dx + \int_{\mathbb{R}^2} \frac{v^2}{(1+|x|^2)^\beta} dx < +\infty\}.$$

We denote the radial subspace of  $\mathcal{D}$  by  $\mathcal{D}_r$  and  $\mathcal{D}_r^2 = \mathcal{D}_r \times \mathcal{D}_r$ . we define a map:

$$G(\epsilon, v_1, v_2) = (G_1(\epsilon, v_1, v_2), G_2(\epsilon, v_1, v_2)) : \mathcal{D}_r^2 \rightarrow \mathcal{D}_r^2$$

for  $t \in [0, 1]$  by

$$\begin{aligned} G_1(\epsilon, v_1, v_2) &= (-\Delta + \sigma)^{-1} [e^{v_2+h_{2\epsilon}}(1 - e^{v_1+h_{1\epsilon}}) + \sigma v_1 - g_{1\epsilon}], \\ G_2(\epsilon, v_1, v_2) &= (-\Delta + \sigma)^{-1} [e^{v_1+h_{1\epsilon}}(1 - e^{v_2+h_{2\epsilon}}) + \sigma v_2 - g_{2\epsilon}], \end{aligned} \quad (2.7)$$

where  $\sigma = \frac{1}{(1+|x|^2)^\beta}$  and  $g_{i\epsilon} = \frac{4(\epsilon N_i + \beta_i)}{(1+|x|^2)^2}$ ,  $i = 1, 2$ . For  $\epsilon \in [0, 1]$ ,  $G(\epsilon, \cdot, \cdot)$  is a continuous compact operator from  $\mathcal{D}_r^2$  to  $\mathcal{D}_r^2$ .

**The proof for Theorem 1.1:**

Under the assumptions of Lemma 2.3, we have  $|v_1|_{\mathcal{D}_r} + |v_2|_{\mathcal{D}_r} \leq C$  uniformly with respect to  $\epsilon \in [0, 1]$ . One can use Lemma 2.3 and integrate by parts to get it. We omit the details here. Therefore, we can choose a number  $R > 0$  independent of  $\epsilon$  such that if  $I - G(\epsilon, v_1, v_2) = 0$  then  $|v_1|_{\mathcal{D}_r} + |v_2|_{\mathcal{D}_r} < R$ . Set  $\Omega_R = \{(v_1, v_2) \in \mathcal{D}_r \times \mathcal{D}_r \mid |v_1|_{\mathcal{D}_r} + |v_2|_{\mathcal{D}_r} < R\}$ . Therefore the degree  $\deg(I - G(\epsilon, \cdot, \cdot), \Omega_R, 0)$  in  $\mathcal{D}_r \times \mathcal{D}_r$  is well defined for  $\epsilon \in [0, 1]$ . Also,  $I - G(\epsilon, \cdot, \cdot)$  defines a good homotopy. We have  $\deg(I - G(1, v_1, v_2), \Omega_R, 0) = \deg(I - G(0, v_1, v_2), \Omega_R, 0)$ . But  $I - G(0, v_1, v_2) = 0$  is equivalent to

$$\begin{cases} \Delta v_1 + e^{v_2}(1 - e^{v_1}) = 0, \\ \Delta v_2 + e^{v_1}(1 - e^{v_2}) = 0, \end{cases} \quad \text{in } \mathbb{R}^2.$$

It is already known that for radially symmetric solutions of  $I - G(0, v_1, v_2) = 0$ , one has  $\deg(I - G(0, v_1, v_2), \Omega_R, 0) = -1$ , see [12]. This proves Theorem 1.1.

### 3 Generalization of Brezis-Merle's alternative

Before proving Theorem 1.2, we will first generalize the Brezis-Merle's alternative for PDE systems as follows in this section. Consider the following system:

$$\begin{cases} \Delta u_{1n} + V_{1n} e^{u_{2n}} = 0, \\ \Delta u_{2n} + V_{2n} e^{u_{1n}} = 0, \end{cases} \quad \text{in } \Omega \subset \mathbb{R}^2. \quad (3.1)$$

**Lemma 3.1.** Assume  $(u_{1n}, u_{2n})$  is a sequence of solutions of (3.1) satisfying

$$\begin{aligned} |V_{1n}|_{L^\infty(\Omega)} + |V_{2n}|_{L^\infty(\Omega)} &\leq C_1, |e^{u_{1n}}|_{L^1(\Omega)} + |e^{u_{2n}}|_{L^1(\Omega)} \leq C_2, \\ \text{either } \int_{\Omega} |V_{1n}| e^{u_{2n}} dx &\leq \epsilon_1 < 4\pi, \text{ or } \int_{\Omega} |V_{2n}| e^{u_{1n}} dx \leq \epsilon_2 < 4\pi. \end{aligned}$$

Then  $u_{1n}^+, u_{2n}^+$  are uniformly bounded in  $L_{loc}^\infty(\Omega)$ .

*Proof.* Split  $u_{kn} = \bar{u}_{kn} + \tilde{u}_{kn}$ ,  $k = 1, 2$  such that

$$\begin{cases} \Delta \bar{u}_{1n} + V_{1n}e^{u_{2n}} = 0, \\ \Delta \bar{u}_{2n} + V_{2n}e^{u_{1n}} = 0, \quad \text{in } \Omega \\ \bar{u}_{1n} = \bar{u}_{2n} = 0 \text{ on } \partial\Omega. \end{cases} \quad (3.2)$$

Now if  $\int_{\Omega} |V_{1n}|e^{u_{2n}} dx \leq \epsilon_1 < 4\pi$ , by Brezis-Merle's inequality, we know that  $\int_{\Omega} e^{(1+\delta)|\bar{u}_{1n}|} dx \leq C$  for some small  $\delta > 0$  and also  $\bar{u}_{1n}, \bar{u}_{2n}$  are uniformly bounded in  $L^1(\Omega)$ . Noting that  $\tilde{u}_{kn}$  is harmonic in  $\Omega$  and

$$|\tilde{u}_{kn}^+|_{L^1(\Omega)} \leq |u_{kn}^+|_{L^1(\Omega)} + |\bar{u}_{kn}|_{L^1(\Omega)} \leq C,$$

we will have  $\tilde{u}_{kn}^+$  is bounded in  $L_{loc}^{\infty}(\Omega)$  by the mean value property of harmonic functions. This means that  $e^{u_{1n}}$  is bounded in  $L_{loc}^{1+\delta}(\Omega)$ . Applying the standard elliptic estimates to  $\bar{u}_{2n}$  yields that  $\bar{u}_{2n}$  is bounded in  $W_{loc}^{2,\gamma}(\Omega)$  for some  $\gamma > 1$ , i.e.  $\bar{u}_{2n}$  is bounded in  $L_{loc}^{\infty}(\Omega)$ . Taking this estimate back to the equation of  $\bar{u}_{1n}$ , we must have  $\bar{u}_{1n}$  is bounded in  $L_{loc}^{\infty}(\Omega)$ . This proves the present lemma.  $\square$

Define the blow-up set  $S$  as follows.

$$S = \{x \in \Omega; \quad \text{there exist two sequences } x_{1n}, x_{2n} \text{ such that } x_{1n}, x_{2n} \rightarrow x, \\ u_{1n}(x_{1n}), u_{2n}(x_{2n}) \rightarrow +\infty\}.$$

**Theorem 3.1.** Assume  $(u_{1n}, u_{2n})$  is a sequence of solutions of (3.1) with

$$|V_{1n}|_{L^{\infty}(\Omega)} + |V_{2n}|_{L^{\infty}(\Omega)} \leq C_1, |e^{u_{1n}}|_{L^1(\Omega)} + |e^{u_{2n}}|_{L^1(\Omega)} \leq C_2$$

Then there exists a subsequence of  $(u_{1n}, u_{2n})$  satisfying that

- (i).  $S = \emptyset$ . Then for each component  $u_{kn}$ , it either is bounded in  $L_{loc}^{\infty}(\Omega)$  or locally uniformly converges to  $-\infty$ .
- (ii).  $S \neq \emptyset$ . Then  $\forall x \in S$ ,  $\exists x_{1n}, x_{2n} \rightarrow x, u_{1n}(x_{1n}), u_{2n}(x_{2n}) \rightarrow +\infty$ . Moreover,  $\forall K \subset \subset \Omega \setminus S$ ,  $u_{1n}(x_{1n}), u_{2n}(x_{2n}) \rightarrow -\infty$  uniformly on  $K$  and

$$V_{1n}e^{u_{2n}} \rightarrow \sum_{r \in S} a_{1r} \delta_r, V_{2n}e^{u_{1n}} \rightarrow \sum_{r \in S} a_{2r} \delta_r, a_{1r}, a_{2r} \geq 4\pi$$

in the sense of measure and  $S$  is a finite set.

*Proof.* Set  $V_{1n}e^{u_{2n}} \rightarrow \mu_1, V_{2n}e^{u_{1n}} \rightarrow \mu_2$  in measure in  $\Omega$ . Define

$$\Sigma = \{x \in \Omega | \mu_1(\{x\}) \geq 4\pi, \mu_2(\{x\}) \geq 4\pi\}.$$

Step 1.  $S = \Sigma$ . First  $S \subset \Sigma$ . If  $x_0 \notin \Sigma$ , without loss of generality, we may choose  $\delta_0$  small enough such that  $\int_{B_{\delta_0}(x_0)} V_{1n}e^{u_{2n}} < 4\pi$ . Applying Lemma 3.1 in  $B_{\delta_0}(x_0)$ , one can get

$$u_{1n}^+, u_{2n}^+ \text{ are bounded in } L_{loc}^{\infty}(B_{\delta_0}(x_0))$$

which means  $x_0 \notin S$ . This proves  $S \subset \Sigma$ . Picking up  $x_0 \in \Sigma$ , we claim that

$$\forall R > 0, \lim_{n \rightarrow \infty} \inf |u_{1n}^+|_{L^{\infty}(B_R(x_0))} \rightarrow +\infty, \lim_{n \rightarrow \infty} \inf |u_{2n}^+|_{L^{\infty}(B_R(x_0))} \rightarrow +\infty.$$

Suppose not, we may assume that

$$|u_{1n}^+|_{L^{\infty}(B_{R_0}(x_0))} \leq C, \quad \text{for some } R_0, \quad \text{uniformly for some constant } C.$$



Then by Hölder's inequality, we can take  $R < R_0$  small enough such that  $\int_{B_R(x_0)} V_{2n} e^{u_{1n}} dx < 4\pi$ . This contradicts with  $x_0 \in \Sigma$  and proves the claim. Set

$$u_{1n}(x_{1n}) = \max_{B_R(x_0)} u_{1n}, u_{2n}(x_{2n}) = \max_{B_R(x_0)} u_{2n}, B_R(x_0) \cap \Sigma = \{x_0\}.$$

Then  $u_{1n}(x_{1n}), u_{2n}(x_{2n}) \rightarrow +\infty$ ,  $x_{1n} \rightarrow x_1, x_{2n} \rightarrow x_2$ . We need to show  $x_1 = x_2 = x_0$ . If not, we may assume  $x_1 \neq x_0$ . By the choice of  $B_R(x_0)$ , we must have  $x_1 \notin \Sigma$  which means  $u_{1n}^+, u_{2n}^+$  are bounded in  $L^\infty(B_\delta(x_1))$  for some small  $\delta > 0$ . This yields a contradiction and  $x_0 \in S$ . This proves  $S = \Sigma$  and  $S$  is a finite set.

Step 2.  $S = \emptyset$  means (i) holds. First if  $S = \emptyset$ , we will have  $u_{1n}^+, u_{2n}^+ \in L_{loc}^\infty(\Omega)$ . If not, we may assume that  $u_{1n}(x_{1n}) \rightarrow +\infty, x_{1n} \rightarrow x_1 \in \Omega$ . We claim that:

$$\exists x_{2n} \rightarrow x_1 \text{ such that } u_{2n}(x_{2n}) \rightarrow +\infty.$$

If not, there exists  $\delta_0 > 0$  such that  $u_{2n}^+$  is bounded in  $L^\infty(B_{\delta_0}(x_1))$ . Repeating the proof in Lemma 3.1, one can get  $u_{1n}^+$  is bounded in  $L_{loc}^\infty(B_{\delta_0}(x_1))$  contradicts to our assumption. This implies  $x_1 \in S$  contradicts to our assumption again. Hence we have  $u_{1n}^+, u_{2n}^+$  are bounded in  $L_{loc}^\infty(\Omega)$ . Then we can apply Harnack inequality to (3.1) to get (i) holds.

Step 3.  $S \neq \emptyset$  implies (ii) holds. By Lemma 3.1, we know that  $u_{1n}^+, u_{2n}^+$  are bounded in  $L_{loc}^\infty(\Omega \setminus S)$ , which means  $V_{1n} e^{u_{2n}} \in L_{loc}^\infty(\Omega \setminus S), V_{2n} e^{u_{1n}} \in L_{loc}^\infty(\Omega \setminus S)$ . This implies  $\mu_1, \mu_2$  are bounded measure on  $\Omega$  with  $\mu_1 \in L_{loc}^\infty(\Omega \setminus S), \mu_2 \in L_{loc}^\infty(\Omega \setminus S)$ . With  $\bar{u}_{kn}, \tilde{u}_{kn}, k = 1, 2$  defined as in Lemma 3.1, we have  $\bar{u}_{1n}, \bar{u}_{2n} \rightarrow \bar{u}_1, \bar{u}_2$  locally uniformly in  $\Omega \setminus S$ . Also by mean value property, we have  $\tilde{u}_{kn}^+ \leq C$ . Applying the Harnack inequality yields

(a). At least one component of  $(\tilde{u}_{1n}, \tilde{u}_{2n})$  is bounded in  $L_{loc}^\infty(\Omega \setminus S)$ .

(b).  $\tilde{u}_{1n}, \tilde{u}_{2n} \rightarrow -\infty$  locally uniformly in  $\Omega \setminus S$ .

We exclude situation (a) as follows. If (a) happens, we may assume  $\tilde{u}_{1n} \in L_{loc}^\infty(\Omega \setminus S)$ . Consider  $x_0 \in S$ . Then for small  $R \leq R_0$ ,  $\tilde{u}_{1n} \in L^\infty(\partial B_R(x_0))$ ,  $|u_{1n}|_{L^\infty(\partial B_R(x_0))} \leq C$ . Consider the following boundary value problem:

$$\begin{cases} -\Delta h_{1n} = V_{1n} e^{u_{2n}}, & \text{in } B_R(x_0), \\ h_{1n} = -C, & \text{in } \partial B_R(x_0). \end{cases}$$

Then by the maximal principle, we have

$$u_{1n} \geq h_{1n}, \quad \text{in } B_R(x_0).$$

In particular  $\int_{B_R(x_0)} e^{h_{1n}} dx \leq \int_{B_R(x_0)} e^{u_{1n}} dx \leq C < \infty$ . On the other hand, we have  $h_{1n} \rightarrow h_1 \in W_{loc}^{2,q}(\bar{B}_R(x_0) \setminus \{0\}), \forall q < \infty$  with  $h_1$  solves

$$\begin{cases} -\Delta h_1 = \mu_1, & \text{in } B_R(x_0), \\ h_1 = -C, & \text{on } \partial B_R(x_0). \end{cases}$$

As  $x_0 \in S$ , we have  $\mu_1\{x_0\} \geq 4\pi$  which implies  $\mu_1 \geq 4\pi\delta_{x_0}$ . One gets that

$$h_1(x) \geq -2 \ln |x - x_0| + O(1).$$

Then  $\int_{B_R(x_0)} e^{h_1} dx = \infty$  yields a contradiction. Thus we must have situation (b) happens. This ends the proof of our theorem.  $\square$

**Remark 3.1.** When we apply Theorem 3.1 to system (1.13), we will obtain  $e^{u_{1n}}, e^{u_{2n}}$  are uniformly bounded in  $L^1(\mathbb{R}^2)$  by Pohozaev's identity even though we only have  $V_{1n} e^{u_{2n}}, V_{2n} e^{u_{1n}}$  are uniformly bounded in  $L^1(\mathbb{R}^2)$ . This will be found in Lemma 4.1.

## 4 A priori estimates

In this section, we will obtain the a priori estimates for the solutions of the following problem

$$\begin{cases} \Delta u_1 + e^{u_2}(1 - e^{u_1}) = 4\pi \sum_{i=1}^{N_1} \delta_{\epsilon p_{1i}}, \\ \Delta u_2 + e^{u_1}(1 - e^{u_2}) = 4\pi \sum_{i=1}^{N_2} \delta_{\epsilon p_{2i}}, \end{cases} \quad \text{in } \mathbb{R}^2 \quad (4.1)$$

where  $\epsilon \in [0, 1]$  with

$$u_i(x) = -2\beta_i \ln |x| + O(1), \quad \beta_i > 1, \quad i = 1, 2, \quad \text{near } \infty. \quad (4.2)$$

and

$$(\beta_1 - 1)(\beta_2 - 1) > (N_1 + 1)(N_2 + 1) \quad (4.3)$$

by blow-up analysis. In what follows we always assume  $\max(N_1, N_2) \geq 1$ . Otherwise, this is the 0-vortex case which has been discussed in [12]. Now suppose  $(u_1, u_2)$  is a solution of system (4.1), (4.2), (4.3). Then applying the maximum principle, we have  $u_1, u_2 < 0$ ,  $\forall x \in \mathbb{R}^2$ . Write

$$u_1(x) = v_1(x) + f_{1\epsilon}(x), u_2(x) = v_2(x) + f_{2\epsilon}(x)$$

where  $f_{1\epsilon}(x) = 2 \sum_{i=1}^{N_1} \ln |x - \epsilon p_{1i}|$ ,  $f_{2\epsilon}(x) = 2 \sum_{i=1}^{N_2} \ln |x - \epsilon p_{2i}|$ . At first, we shall establish the following Pohozaev's identities.

**Lemma 4.1.** *Let  $(u_1, u_2)$  be a solution of (4.1), (4.2), (4.3). Then  $(u_1, u_2)$  satisfies*

$$\begin{aligned} \int_{\mathbb{R}^2} e^{u_1} dx &= 4\pi(\beta_1\beta_2 - N_1N_2 - \beta_1 - N_1) - 2\pi \left( \sum_{i=1}^{N_1} \epsilon p_{1i} \cdot \nabla v_2(\epsilon p_{1i}) + \sum_{i=1}^{N_2} \epsilon p_{2i} \cdot \nabla v_1(\epsilon p_{2i}) \right) \\ \int_{\mathbb{R}^2} e^{u_2} dx &= 4\pi(\beta_1\beta_2 - N_1N_2 - \beta_2 - N_2) - 2\pi \left( \sum_{i=1}^{N_1} \epsilon p_{1i} \cdot \nabla v_2(\epsilon p_{1i}) + \sum_{i=1}^{N_2} \epsilon p_{2i} \cdot \nabla v_1(\epsilon p_{2i}) \right) \\ \int_{\mathbb{R}^2} e^{u_1+u_2} dx &= 4\pi((\beta_1 - 1)(\beta_2 - 1) - (N_1 + 1)(N_2 + 1)) - 2\pi \sum_{i=1}^{N_1} \epsilon p_{1i} \cdot \nabla v_2(\epsilon p_{1i}) \\ &\quad - 2\pi \sum_{i=1}^{N_2} \epsilon p_{2i} \cdot \nabla v_1(\epsilon p_{2i}) \end{aligned}$$

*Proof.* In terms of  $(v_1, v_2)$ , (4.1) becomes

$$\begin{cases} \Delta v_1 + e^{u_2}(1 - e^{u_1}) = 0, \\ \Delta v_2 + e^{u_1}(1 - e^{u_2}) = 0. \end{cases}$$

Multiplying the first equation by  $x \cdot \nabla(v_2 + f_{2\epsilon})$ , the second equation by  $x \cdot \nabla(v_1 + f_{1\epsilon})$ , integrating over  $B_R$  and summing them up, we get

$$\begin{aligned} & \int_{\partial B_R} (x \cdot \nu)(\nabla v_1 \cdot \nabla v_2) dS - \int_{\partial B_R} (x \cdot \nabla v_1)(\nu \cdot \nabla v_2) dS - \int_{\partial B_R} (x \cdot \nabla v_2)(\nabla v_1 \cdot \nu) dS \\ &= \int_{\partial B_R} (x \cdot \nu)(e^{u_1} + e^{u_2} - e^{u_1+u_2}) dS - 2 \int_{B_R} (e^{u_1} + e^{u_2} - e^{u_1+u_2}) dx \end{aligned}$$

$$+ \int_{B_R} (x \cdot \nabla f_{1\epsilon}) \Delta v_2 dx + \int_{B_R} (x \cdot \nabla f_{2\epsilon}) \Delta v_1 dx$$

with  $\nu = \frac{x}{|x|}$ . Since  $v_i(x) = -2(\beta_i + N_i) \ln |x| + O(1)$  near  $\infty$ , we can get

$$LHS = -8\pi(\beta_1 + N_1)(\beta_2 + N_2) + o(1).$$

We need to estimate the last two terms on the righthand side,

$$\begin{aligned} \int_{B_R} (x \cdot \nabla f_{1\epsilon}) \Delta v_2 dx &= \int_{B_R} \sum_{i=1}^{N_1} \frac{2x \cdot (x - \epsilon p_{1i})}{|x - \epsilon p_{1i}|^2} \Delta v_2 dx \\ &= -8\pi N_1(\beta_2 + N_2) + \int_{B_R} \sum_{i=1}^{N_1} 2\epsilon p_{1i} \cdot \nabla \ln |x - \epsilon p_{1i}| \Delta v_2 dx \\ &= -8\pi N_1(\beta_2 + N_2) - 4\pi \sum_{i=1}^{N_1} \epsilon p_{1i} \cdot \nabla v_2(\epsilon p_{1i}) + o(1). \end{aligned}$$

In getting the last equality, we have used the fact that  $\ln |x - \epsilon p_{1i}|$  is the Green function and the decay property of  $|\nabla^k v_2|$ ,  $k = 1, 2$  at  $\infty$ . Also one can note that

$$\int_{\mathbb{R}^2} e^{u_2} (1 - e^{u_1}) dx = 4\pi(\beta_1 + N_1), \quad \int_{\mathbb{R}^2} e^{u_1} (1 - e^{u_2}) dx = 4\pi(\beta_2 + N_2).$$

Combining all the estimates above, we can get the desired identities.  $\square$

Our main job in this section is to prove the following theorem.

**Theorem 4.1.** *Let  $p_{11}, \dots, p_{1N_1}, p_{21}, \dots, p_{2N_2}$  be given. For any  $(\beta_1, \beta_2)$  satisfying (4.3) and*

$$\frac{N_1}{\beta_1 + N_1} + \frac{N_2}{\beta_2 + N_2} \notin \left\{ \frac{k-1}{k} \mid k = 2, \dots, \max(N_1, N_2) \right\}. \quad (4.4)$$

*for any compact set  $K$ , there exists a constant  $C = C(\beta_1, \beta_2, N_1, N_2, \max(|p_{ij}|), K)$  independent of  $\epsilon$  such that for any solution  $(u_1, u_2)$  of (4.1), (4.2),*

$$|u_1 - f_{1\epsilon}|_{L^\infty(K)} + |u_2 - f_{2\epsilon}|_{L^\infty(K)} \leq C. \quad (4.5)$$

We will prove Theorem 4.1 by contradiction. If Theorem 4.1 is not true. Then we have for some compact set  $K$  and  $\epsilon_n \rightarrow \epsilon^* \in [0, 1]$  such that

$$|u_{1n} - f_{1n}|_{L^\infty(K)} + |u_{2n} - f_{2n}|_{L^\infty(K)} \rightarrow \infty, \text{ as } n \rightarrow \infty,$$

where  $f_{in} = \sum_{j=1}^{N_i} 2 \ln |x - \epsilon_n p_{ij}|$ ,  $i = 1, 2$ . Recall that for  $v_{in}(x) = u_{in}(x) - f_{in}(x)$ ,  $i = 1, 2$ ,

$$\begin{cases} \Delta v_{1n} + e^{u_{2n}} (1 - e^{u_{1n}}) = 0, \\ \Delta v_{2n} + e^{u_{1n}} (1 - e^{u_{2n}}) = 0, \end{cases} \quad \text{in } \mathbb{R}^2.$$

From  $u_{1n}, u_{2n} < 0$  in  $\mathbb{R}^2$ , we know that for some fixed large  $R > \max |p_{ij}|$ ,  $K \subset \subset B_R$ ,

$$\Delta(v_{1n} + \frac{1}{2}|x|^2) \geq 0, \Delta(v_{2n} + \frac{1}{2}|x|^2) \geq 0, \text{ in } B_R.$$

This implies that

$$\max_{B_R} (v_{in} + \frac{1}{2}|x|^2) \leq \max_{\partial B_R} (v_{in} + \frac{1}{2}|x|^2) \leq \max_{\partial B_R} -f_{in} + \frac{1}{2}R^2 \leq C.$$

By this, one can get  $v_{1n}, v_{2n}$  are uniformly upper bounded in  $B_R$ . This allows us to assume that

$$\min_{|x| \leq R} v_{1n} \rightarrow -\infty.$$

By Harnack inequality, we will have  $v_{1n} \rightarrow -\infty$  locally uniformly in  $\mathbb{R}^2$ , also  $u_{1n} \rightarrow -\infty$  locally uniformly in  $\mathbb{R}^2$  as  $n \rightarrow \infty$ .

**Lemma 4.2.** *If  $v_{1n}$  blows up in a compact set  $K$ , then  $v_{2n}$  blows up in the compact set  $K$  too.*

*Proof.* If not, we may assume  $|v_{2n}|_{L^\infty(K)} \leq C$ . By Harnack inequality and standard elliptic estimates, one can get that  $v_{2n}(x) \rightarrow v_2(x)$  uniformly in  $C_{loc}^2(\mathbb{R}^2)$  with

$$\Delta v_2 = 0, \text{ in } \mathbb{R}^2$$

as we notice that  $u_{1n} \rightarrow -\infty$  locally uniformly. This implies that  $u_{2n} \rightarrow u_2$  uniformly in  $C_{loc}^2(\mathbb{R}^2 \setminus \{\epsilon^* p_{2j}, j = 1, \dots, N_2\})$  with

$$\Delta u_2 = 4\pi \sum_{j=1}^{N_2} \delta_{\epsilon^* p_{2j}}, \text{ in } \mathbb{R}^2.$$

By Lemma 4.1, we need to estimate  $\nabla v_{1n}(\epsilon_n p_{1j}), \nabla v_{2n}(\epsilon_n p_{2j})$  to get the uniform bound of  $\int_{\mathbb{R}^2} e^{u_{2n}} dx$ .

By Green's representation formula for  $v_{in}, i = 1, 2$ , we have

$$v_{1n}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{|y|}{|x-y|} e^{u_{2n}(y)} (1 - e^{u_{1n}(y)}) dy + c_{1n}, \text{ for some constant } c_{1n}.$$

From this, one gets

$$\begin{aligned} |\nabla v_{1n}(x)| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|x-y|} e^{u_{2n}(y)} (1 - e^{u_{1n}(y)}) dy \\ &\leq \frac{1}{2\pi} \int_{|y-x| \leq 1} \frac{1}{|x-y|} dy + \frac{1}{2\pi} \int_{|x-y| \geq 1} e^{u_{2n}(y)} (1 - e^{u_{1n}(y)}) dy \\ &\leq C_1 + C_2 \leq C < \infty, \text{ independent of } n. \end{aligned}$$

The argument for  $|\nabla v_{2n}|$  is just the same. This indicates that  $\int_{\mathbb{R}^2} e^{u_{2n}} dx$  is uniformly bounded. As  $u_2$  is harmonic in  $\mathbb{R}^2 \setminus B_{R_0}$ , for  $R_0 \geq \max |p_{2j}| + 1$ , we get

$$\bar{u}_2(R) = \frac{1}{2\pi} \int_{\partial B_R} u_2 dS = \bar{u}_2(R_0), R \geq R_0.$$

Now we have

$$\int_{|x| \geq R_0} e^{u_2} dx \geq \int_{R_0}^{\infty} 2\pi r^2 e^{\bar{u}_2(r)} dr = +\infty$$

which contradicts to

$$\int_{\mathbb{R}^2} e^{u_2} dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} e^{u_{2n}} dx < \infty.$$

This means  $v_{2n}$  must blow up simultaneously.  $\square$

**Lemma 4.3.** *Let  $x_{in}$  be the maximum point of  $u_{in}, i = 1, 2$  respectively. Then  $|x_{in}| \rightarrow \infty$  as  $n \rightarrow \infty$  and  $u_{in}(x_{in}) \geq -C$  uniformly for some constant  $C > 0$ .*

*Proof.* By the previous proof in Lemma 4.2, we know that  $u_{1n}, u_{2n} \rightarrow -\infty$  locally uniformly in  $\mathbb{R}^2$ . Suppose  $x_{1n}$  is uniformly bounded. Then we will have  $u_{1n}(x) \rightarrow -\infty$  uniformly in  $\mathbb{R}^2$ . Also by Green's representation formula, we have for  $x \in K \subset \subset \mathbb{R}^2$ ,

$$\begin{aligned} |\nabla v_{1n}(x)| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|x-y|} e^{u_{2n}(y)} (1 - e^{u_{1n}(y)}) dy \\ &\leq \frac{1}{2\pi} \int_{|y-x| \leq R} \frac{1}{|x-y|} e^{u_{2n}(y)} (1 - e^{u_{1n}(y)}) dy + \frac{1}{2\pi R} \int_{|x-y| \geq R} e^{u_{2n}(y)} (1 - e^{u_{1n}(y)}) dy \\ &\leq o(1)R + O(R^{-1}). \end{aligned}$$

Let  $n \rightarrow \infty$ , then  $R \rightarrow \infty$ , one has  $\nabla v_{1n} \rightarrow 0$  locally uniformly in  $\mathbb{R}^2$ . The same estimate also holds for  $\nabla v_{2n}$ . Then by Lemma 4.1, we have

$$4\pi[(\beta_1 - 1)(\beta_2 - 1) - (N_1 + 1)(N_2 + 1)] + o(1) = \int_{\mathbb{R}^2} e^{u_{1n} + u_{2n}} dx \leq e^{u_{1n}(x_{1n})} \int_{\mathbb{R}^2} e^{u_{2n}} dx \rightarrow 0$$

which contradicts to  $(\beta_1 - 1)(\beta_2 - 1) - (N_1 + 1)(N_2 + 1) \geq c_0 > 0$ . This implies  $x_{in} \rightarrow \infty$ . We also note that by Lemma 4.1,

$$\int_{\mathbb{R}^2} e^{u_{1n} + u_{2n}} dx \leq 4\pi e^{u_{1n}(x_{1n})} (\beta_1 \beta_2 - N_1 N_2 - \beta_2 - N_2) + o(1) \leq C_0 > 0$$

which means  $\max_{\mathbb{R}^2} u_{1n} \geq -C$  uniformly. The same conclusion holds also for  $u_{2n}$ .  $\square$

Now by Lemma 4.3, we set  $r_n = |x_{1n}|^{-1}$ , then  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ . Set

$$w_{in}(x) = u_{in} \left( \frac{x}{r_n} \right) - 2 \ln r_n. \quad (4.6)$$

Then  $w_{in}$  satisfies

$$\begin{cases} \Delta w_{1n} + e^{w_{2n}} (1 - r_n^2 e^{w_{1n}}) = 4\pi \sum_{i=1}^{N_1} \delta_{r_n \epsilon_n p_{1i}}, \\ \Delta w_{2n} + e^{w_{1n}} (1 - r_n^2 e^{w_{2n}}) = 4\pi \sum_{i=1}^{N_2} \delta_{r_n \epsilon_n p_{2i}}, \end{cases} \quad \text{in } \mathbb{R}^2. \quad (4.7)$$

Obviously, there holds  $r_n \epsilon_n p_{ij} \rightarrow 0, i = 1, 2$  as  $n \rightarrow \infty$ . By Theorem 3.1 and noting that  $w_{1n}(r_n x_{1n}) \rightarrow +\infty$  which means the blow up case in Theorem 3.1 happens. Since  $\lim_{n \rightarrow \infty} r_n x_{1n} \rightarrow q \in \mathbb{S}^1$ , there exists a non-empty finite set  $S$  of nonzero points such that  $w_{in} \rightarrow -\infty, i = 1, 2$  uniformly on each  $K \subset \subset \mathbb{R}^2 \setminus (S \cup \{0\})$  and

$$e^{w_{2n}} (1 - r_n^2 e^{w_{1n}}) \rightarrow \sum_{q \in S} 2\pi M_q \delta_q, \quad e^{w_{1n}} (1 - r_n^2 e^{w_{2n}}) \rightarrow \sum_{q \in S} 2\pi N_q \delta_q$$

on any  $D \subset \subset \mathbb{R}^2 \setminus \{0\}$  with  $S \subset D$  in the distribution sense. For any  $q \in S$ , set  $d$  small such that  $B_d(q) \cap (S \cup \{0\}) = \{q\}$  and

$$\begin{aligned} M_{q,n} &= \frac{1}{2\pi} \int_{B_d(q)} e^{w_{2n}} (1 - r_n^2 e^{w_{1n}}) dx, \quad M_{q,n} \rightarrow M_q, \quad \text{as } n \rightarrow \infty, \\ N_{q,n} &= \frac{1}{2\pi} \int_{B_d(q)} e^{w_{1n}} (1 - r_n^2 e^{w_{2n}}) dx, \quad N_{q,n} \rightarrow N_q, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Repeating the proof of Lemma 4.1 over  $B_d(q)$  and by Pohozaev's identity, we have

$$\begin{aligned} \int_{B_d(q)} e^{w_{1n}} dx &= \pi M_q (N_q - 2) + o(1), \quad \int_{B_d(q)} e^{w_{2n}} dx = \pi N_q (M_q - 2) + o(1), \\ \int_{B_d(q)} r_n^2 e^{w_{1n} + w_{2n}} dx &= \pi (N_q M_q - 2M_q - 2N_q) + o(1). \end{aligned} \quad (4.8)$$

(4.8) implies that  $M_q, N_q > 2$  and  $N_q M_q - 2M_q - 2N_q \geq 0$ . Note that

$$\left( \frac{M_q + N_q}{2} \right)^2 \geq M_q N_q \geq 2(M_q + N_q).$$

We have  $M_q + N_q \geq 8$  and  $|S| \leq \frac{\beta_1 + \beta_2 + N_1 + N_2}{4}$ . In the following, we will prove that all  $M_q$ 's and  $N_q$ 's are the same respectively. For this purpose, we need to show the local estimates for  $w_{1n}, w_{2n}$  and the simple blow up property of  $w_{1n}, w_{2n}$ . For any  $q \in S$ , for  $d$  small enough

$$w_{in}(q_{in}) = \max_{B_d(q)} w_{in}(x) \rightarrow +\infty, i = 1, 2.$$

We have the following estimates.

**Lemma 4.4.**

$$\max_{|x-q| \leq d} (w_{in}(x) + 2 \ln |x - q_{in}|) \leq C, i = 1, 2.$$

*Proof.* If the lemma is not right, without loss of generality, one may assume that

$$w_{1n}(y_n) + 2 \ln |y_n - q_{1n}| = \max_{|x-q| \leq d} (w_{1n}(x) + 2 \ln |x - q_{1n}|) \rightarrow +\infty.$$

It is easy to see that  $y_n \neq q_{1n}$  and  $q_{1n}, y_n \rightarrow q$ . Set  $d_n = |y_n - q_{1n}|$  and

$$\bar{w}_{in}(x) = w_{in}(d_n x + q_{1n}) + 2 \ln d_n, i = 1, 2, \forall |x| \leq \frac{d}{2d_n}.$$

And  $\bar{w}_{in}$  satisfies that

$$\begin{cases} \Delta \bar{w}_{1n} + e^{\bar{w}_{2n}} (1 - r_n^2 d_n^{-2} e^{\bar{w}_{1n}}) = 0, \\ \Delta \bar{w}_{2n} + e^{\bar{w}_{1n}} (1 - r_n^2 d_n^{-2} e^{\bar{w}_{2n}}) = 0, \end{cases} \quad \text{in } B_{\frac{d}{2d_n}}(0) \quad (4.9)$$

with

$$\begin{aligned} \bar{w}_{1n}(0) &= w_{1n}(q_{1n}) + 2 \ln d_n \geq w_{1n}(y_n) + 2 \ln d_n \rightarrow +\infty, \\ \bar{w}_{1n} \left( \frac{y_n - q_{1n}}{|y_n - q_{1n}|} \right) &= w_{1n}(y_n) + 2 \ln d_n \rightarrow +\infty. \end{aligned}$$

Hence,  $\bar{w}_{1n}$  blows up at 0 and some point  $e = \lim_{n \rightarrow +\infty} \frac{y_n - q_{1n}}{|y_n - q_{1n}|}$ . By Theorem 3.1,  $\bar{w}_{1n}, \bar{w}_{2n}$  must blow up simultaneously at these two different points. Denote the blow up set of  $\bar{w}_{1n}, \bar{w}_{2n}$  by  $S^* = \{z_1, \dots, z_l\}$ ,  $l \geq 2$ . We have

$$e^{\bar{w}_{2n}} (1 - r_n^2 d_n^{-2} e^{\bar{w}_{1n}}) \rightarrow \sum_{i=1}^l 2\pi m_i \delta_{z_i}, \quad e^{\bar{w}_{1n}} (1 - r_n^2 d_n^{-2} e^{\bar{w}_{2n}}) \rightarrow \sum_{i=1}^l 2\pi n_i \delta_{z_i}.$$

Consider a unit vector  $\xi \in \mathbb{R}^2$ . Multiplying the first equation of (4.9) by  $\xi \cdot \nabla \bar{w}_{2n}$  and the second equation by  $\xi \cdot \nabla \bar{w}_{1n}$ , integrating by parts in  $B_d(z_k)$ , we get the following Pohozaev's identity,

$$\begin{aligned} & \int_{\partial B_d(z_k)} (\xi \cdot \nabla \bar{w}_{2n})(\nu \cdot \nabla \bar{w}_{1n}) dS + \int_{\partial B_d(z_k)} (\nu \cdot \nabla \bar{w}_{2n})(\xi \cdot \nabla \bar{w}_{1n}) dS \\ &= \int_{\partial B_d(z_k)} (\xi \cdot \nu)(\nabla \bar{w}_{2n} \cdot \nabla \bar{w}_{1n}) dS + \int_{\partial B_d(z_k)} (\xi \cdot \nu) \left( \frac{r_n^2}{d_n^2} e^{\bar{w}_{1n} + \bar{w}_{2n}} - e^{\bar{w}_{1n}} - e^{\bar{w}_{2n}} \right) dS \end{aligned} \quad (4.10)$$

Here  $\nu = \frac{x-z_k}{|x-z_k|}$ ,  $B_d(z_k) \cap S^* = z_k$ . Using Green's representation formula of  $u_{1n}, u_{2n}$ , we have for any  $x \in K \subset \mathbb{R}^2 \setminus S^*$ , for some fixed  $p_0 \in \mathbb{R}^2 \setminus S^*$ ,

$$\begin{aligned} & \bar{w}_{1n}(x) - \bar{w}_{1n}(p_0) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \left| \frac{p_0 - y}{x - y} \right| e^{\bar{w}_{2n}} \left( 1 - \frac{r_n^2}{d_n^2} e^{\bar{w}_{1n}(y)} \right) dy + 2 \sum_{i=1}^{N_1} \ln \left| \frac{d_n x + q_{1n} - r_n \epsilon_n p_{1i}}{d_n p_0 + q_{1n} - r_n \epsilon_n p_{1i}} \right| \\ &= I_1 + I_2. \end{aligned}$$

Since  $q_{1n} \rightarrow q \neq 0, r_n \epsilon_n p_{1i} \rightarrow 0$  as  $n \rightarrow \infty$ , it is easy to find that  $I_2 \rightarrow 0$  uniformly on  $K$ . Split the integral domain of  $I_1$  into  $\cup_{z_k \in S^*} B_r(z_k)$ ,  $B_R(0) \setminus \cup_{z_k \in S^*} B_r(z_k)$  and  $\mathbb{R}^2 \setminus B_R(0)$  and denote the corresponding integrals by  $J_1, J_2, J_3$  respectively. Obviously,

$$J_1 \rightarrow \sum_{i=1}^l m_i \ln \left| \frac{p_0 - z_i}{x - z_i} \right| \quad \text{uniformly in } K.$$

As for  $J_2$ , noting that  $\bar{w}_{2n} \rightarrow -\infty$  uniformly in  $B_R(0) \setminus \cup_{z_k \in S^*} B_r(z_k)$ , we have  $J_2 \rightarrow 0$ . For  $J_3$ , we have  $y \in \mathbb{R}^2 \setminus B_R(0)$ ,  $x \in K$ ,

$$1 - c_1 R^{-1} \leq \left| \frac{p_0 - y}{x - y} \right| \leq 1 + c_2 R^{-1}, \quad \int_{\mathbb{R}^2} e^{\bar{w}_{2n}} \left( 1 - \frac{r_n^2}{d_n^2} e^{\bar{w}_{1n}(y)} \right) dy \leq C$$

for some constants  $c_1, c_2, C$  independent of  $R, n$ . This implies  $J_3 \rightarrow O(R^{-1})$ . Let  $n \rightarrow \infty$  then  $R \rightarrow \infty$ . This yields immediately that

$$\bar{w}_{1n}(x) - \bar{w}_{1n}(p_0) \rightarrow \sum_{i=1}^l m_i \ln \left| \frac{p_0 - z_i}{x - z_i} \right| \quad \text{uniformly in } K.$$

The same arguments also lead to

$$\bar{w}_{2n}(x) - \bar{w}_{2n}(p_0) \rightarrow \sum_{i=1}^l n_i \ln \left| \frac{p_0 - z_i}{x - z_i} \right| \quad \text{uniformly in } K.$$

Set

$$H_k(x) = \sum_{j \neq k} m_j \ln \left| \frac{p_0 - z_j}{x - z_j} \right|, \quad H_k^*(x) = \sum_{j \neq k} n_j \ln \left| \frac{p_0 - z_j}{x - z_j} \right|.$$

Using Pohozaev's identity (4.10) and the fact  $\int_{\partial B_d(z_k)} \xi \cdot \nu dS = 0$ , we can get

$$\begin{aligned} & -\frac{1}{d} \int_{\partial B_d(z_k)} \xi \cdot (m_k \nabla H_k^* + n_k \nabla H_k) dS + \int_{\partial B_d(z_k)} (\xi \cdot \nabla H_k^*)(\nu \cdot \nabla H_k) + (\xi \cdot \nabla H_k)(\nu \cdot \nabla H_k^*) dx \\ &= \int_{\partial B_d(z_k)} (\xi \cdot \nu)(\nabla H_k^* \cdot \nabla H_k) dx + \int_{\partial B_d(z_k)} (\xi \cdot \nu) \left( \frac{r_n^2}{d_n^2} e^{\bar{w}_{1n} + \bar{w}_{2n}} - e^{\bar{w}_{1n}} - e^{\bar{w}_{2n}} \right) dS + o(1). \end{aligned}$$

Let  $n \rightarrow \infty$  and  $d \rightarrow 0$ . By noting the arbitrariness of  $\xi$ , then we get

$$m_k \nabla H_k^*(z_k) + n_k \nabla H_k(z_k) = 0, k = 1, \dots, l. \quad (4.11)$$

As  $l \geq 2$  and  $0, e \in S^*$ , without loss of generality, we may assume  $z_{1,1} = \max_{k \leq l} z_{k,1}$  and  $z_{1,1} > z_{2,1}$  where  $z_{k,1}$  stands for the first coordinate of  $z_k$ . By this choice, we can get

$$\sum_{j \neq 1} \frac{(m_1 n_j + n_1 m_j)(z_{1,1} - z_{j,1})}{|z_1 - z_j|^2} > 0$$

which contradicts to (4.11) with  $k = 1$ . This completes the proof of the present lemma.  $\square$

Moreover, we also have the following lemma.

**Lemma 4.5.**

$$\max_{|x-q|\leq d} (w_{1n}(x) + 2 \ln |x - q_{2n}|) \leq C, \quad \max_{|x-q|\leq d} (w_{2n}(x) + 2 \ln |x - q_{1n}|) \leq C.$$

*Proof.* The same as the proof in Lemma 4.4, we obtain the conclusion by contradiction. Suppose not, we have

$$w_{1n}(y_n) + 2 \ln |y_n - q_{2n}| = \max_{|x-q|\leq d} (w_{1n}(x) + 2 \ln |x - q_{2n}|) \rightarrow +\infty.$$

It is obvious that  $y_n \rightarrow q$  as  $n \rightarrow \infty$ . Set

$$\bar{w}_{in}(x) = w_{in}(d_n x + q_{2n}) + 2 \ln d_n, i = 1, 2, \forall |x| \leq \frac{d}{2d_n}$$

where  $d_n = |y_n - q_{2n}|$ .  $\bar{w}_{in}(x)$  satisfies the same equation as (4.9). Note that

$$\bar{w}_{1n} \left( \frac{y_n - q_{2n}}{|y_n - q_{2n}|} \right) = w_{1n}(y_n) + 2 \ln d_n \rightarrow +\infty.$$

Theorem 3.1 tells us that the blow-up case happens. Now we need to prove the blow up set contains at least two different points. Denote

$$\lim_{n \rightarrow \infty} \frac{y_n - q_{2n}}{|y_n - q_{2n}|} = q^* \in \mathbb{S}^1, |q^*| = 1.$$

By Theorem 3.1, there exists a sequence of points  $z_n \rightarrow q^*$ , such that  $\bar{w}_{2n}(z_n) \rightarrow \infty$  and  $|z_n| \geq \frac{1}{2}$ . Since

$$\bar{w}_{2n}(0) = w_{2n}(q_{2n}) + 2 \ln d_n \geq w_{2n}(d_n z_n + q_{2n}) + 2 \ln d_n = \bar{w}_{2n}(z_n) \rightarrow +\infty,$$

one gets there are at least two different blow-up points. Then following the same arguments as in Lemma 4.4, we can prove Lemma 4.5.  $\square$

With the above two lemmas, following the arguments in [1], we now prove the simple blow-up estimates.

**Lemma 4.6.** *For any  $x \in B_d(q), q \in S$ , we have*

$$\begin{aligned} \left| w_{1n}(x) - w_{1n}(q_{1n}) + \frac{M_{q,n}}{2} \ln(1 + e^{w_{1n}(q_{1n})} |x - q_{1n}|^2) \right| &\leq C, \\ \left| w_{2n}(x) - w_{2n}(q_{2n}) + \frac{N_{q,n}}{2} \ln(1 + e^{w_{2n}(q_{2n})} |x - q_{2n}|^2) \right| &\leq C. \end{aligned}$$

*Proof.* Set

$$\bar{w}_{in}(x) = w_{in}(s_n x + q_{1n}) + 2 \ln s_n, \quad s_n = \exp \left( -\frac{w_{1n}(q_{1n})}{2} \right), \quad x \in B_{2d/s_n}(0),$$

where  $B_{4d}(q) \cap S = \{q\}$ . Then  $w_{in}, i = 1, 2$  satisfies the same system as (4.9) with  $d_n$  replaced by  $s_n$ . Since  $\bar{w}_{1n}(0) = 0$  and  $u_{1n}(x) < 0$ , we have  $\frac{r_n}{s_n} \leq 1$ . Also, we can apply Harnack inequality and (i) of Theorem 3.1 in any compact domain  $K$  contains 0 to get that  $\bar{w}_{1n} \in L^\infty(K)$  and either  $\bar{w}_{2n} \in L^\infty(K)$  or  $\bar{w}_{2n} \rightarrow -\infty$  locally uniformly. By the same arguments as in Lemma 4.2, we can exclude the case  $\bar{w}_{2n} \rightarrow -\infty$  if we notice that  $\int e^{\bar{w}_{1n}} dx \leq C < \infty$ . Now we can take a subsequence such that  $\bar{w}_{kn} \rightarrow \bar{w}_k$  in  $C_{loc}^2(\mathbb{R}^2)$  with  $\bar{w}_k$  satisfying

$$\begin{cases} \Delta \bar{w}_1 + e^{\bar{w}_2} (1 - c^2 e^{\bar{w}_1}) = 0, \\ \Delta \bar{w}_2 + e^{\bar{w}_1} (1 - c^2 e^{\bar{w}_2}) = 0, \end{cases} \quad \text{in } \mathbb{R}^2$$



for some constant  $0 \leq c \leq 1$ . By the  $L^1$  integrability of  $e^{\bar{w}_1}, e^{\bar{w}_2}, c^2 e^{\bar{w}_1 + \bar{w}_2}$  in  $\mathbb{R}^2$  and Green's representation formula for  $\bar{w}_1, \bar{w}_2$ , we must have

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{\bar{w}_2} (1 - c^2 e^{\bar{w}_1}) dx > 2, \quad \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{\bar{w}_1} (1 - c^2 e^{\bar{w}_2}) dx > 2.$$

This implies we can choose  $R_0$  large enough and  $\delta_0 > 0$  small enough such that

$$\frac{1}{2\pi} \int_{B_{R_0}} e^{\bar{w}_{2n}} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}}) dx > 2 + 4\delta_0, \quad \frac{1}{2\pi} \int_{B_{R_0}} e^{\bar{w}_{1n}} (1 - r_n^2/s_n^2 e^{\bar{w}_{2n}}) dx > 2 + 4\delta_0.$$

We claim that

$$\bar{w}_{1n}(x) \leq -(2 + \delta_0) \ln |x| + C_0, \text{ for } 2R_0 \leq |x| \leq \frac{d}{s_n}. \quad (4.12)$$

By Green's representation formula and the same arguments as in Lemma 4.4, one can get for  $2R_0 \leq |x| \leq d/s_n$

$$\bar{w}_{1n}(x) = \frac{1}{2\pi} \int_{|y| \leq d/s_n} \ln \frac{|y|}{|x-y|} e^{\bar{w}_{2n}(y)} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}(y)}) dy + O(1)$$

where  $O(1)$  always denotes some uniform bounded term from now on. Split the integral into three domains:  $D_1 = \{|y| \leq R_0\}$ ,  $D_2 = \{|y-x| \leq \frac{|x|}{2}, R_0 \leq |y| \leq d/s_n\}$ ,  $D_3 = \{|y-x| \geq \frac{|x|}{2}, R_0 \leq |y| \leq d/s_n\}$ . In  $D_1$ , we have  $\frac{|x|}{2} \leq |x-y| \leq \frac{3|x|}{2}$ , then

$$\int_{D_1} \ln \frac{|y|}{|x-y|} e^{\bar{w}_{2n}(y)} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}(y)}) dy = O(1) - \int_{|y| \leq R_0} e^{\bar{w}_{2n}(y)} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}(y)}) dy \ln |x|.$$

In  $D_2$ , we have  $\frac{3}{2}|x| \geq |y| \geq \frac{|x|}{2}$ . From Lemma 4.4 and Lemma 4.5, one can get

$$\bar{w}_{kn}(x) + 2 \ln |x| = w_{kn}(z) + 2 \ln |z - q_{1n}| \leq C, \quad z = s_n x + q_{1n}.$$

Substituting this into the integration on  $D_2$ , we have

$$\int_{D_2} \ln \frac{|y|}{|x-y|} e^{\bar{w}_{2n}(y)} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}(y)}) dy \leq O(1) + \frac{C}{|x|^2} \left( \pi (|x|/2)^2 \ln |x| - \pi \int_0^{\frac{|x|}{2}} \ln r dr^2 \right) = O(1)$$

In  $D_3$ , we have  $|x-y| \geq \frac{|y|}{3}$ . This implies

$$\int_{D_3} \ln \frac{|y|}{|x-y|} e^{\bar{w}_{2n}(y)} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}(y)}) dy \leq C.$$

Combining the above three estimates and the choice of  $R_0$ , we prove the claim (4.12). The same estimates also hold true for  $\bar{w}_{2n}$ . By the claim, we have

$$\int_{|y| \leq d/s_n} \ln |y| e^{\bar{w}_{2n}} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}}) dy = O(1)$$

which implies

$$\bar{w}_{1n}(x) = -M_{q,n} \ln |x| + O(1) + \frac{1}{2\pi} \int_{|y| \leq d/s_n} \ln \frac{|x|}{|x-y|} e^{\bar{w}_{2n}} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}}) dy, \quad 2R_0 \leq |x| \leq \frac{d}{s_n}.$$

We now estimate the last term in the above equation. Split the integral into three parts:  $K_1 = \{|y| \leq \frac{|x|}{2}\}$ ,  $K_2 = \{|y-x| \leq \frac{|x|}{2}, |y| \leq d/s_n\}$ ,  $K_3 = \{|y-x| \geq \frac{|x|}{2}, \frac{|x|}{2} \leq |y| \leq d/s_n\}$ . In  $K_1$ , we

have  $2/3 \leq \frac{|x|}{|y-x|} \leq 2$ . Thus the integration on  $K_1$  is  $O(1)$ . In  $K_2$ , we have  $|y| \geq |x|/2 \geq R_0$ , therefore

$$\left| \int_{K_2} \ln \frac{|x|}{|x-y|} e^{\bar{w}_{2n}} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}}) dy \right| \leq c|x|^{-2-\delta_0} \left| \int_{|y-x| \leq |x|/2} \ln \frac{|x|}{|x-y|} dy \right| \leq C.$$

In  $K_3$ , we have  $\frac{1}{2|y|} \leq \frac{|x|}{|y-x|} \leq 2$ , then

$$\left| \int_{K_3} \ln \frac{|x|}{|x-y|} e^{\bar{w}_{2n}} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}}) dy \right| \leq C + \left| \int_{K_3} \ln |y| e^{\bar{w}_{2n}} (1 - r_n^2/s_n^2 e^{\bar{w}_{1n}}) dy \right| \leq C_1.$$

The three estimates yield that

$$\bar{w}_{1n}(x) + M_{q,n} \ln |x| = O(1), \quad \text{for } 2R_0 \leq |x| \leq d/s_n.$$

Combining with the uniform bound of  $\bar{w}_{1n}$  in  $B_{2R_0}$ , we finish the proof of simple blow-up estimates.  $\square$

Now we can prove all the  $M_q$ 's and  $N_q$ 's are the same respectively.

**Lemma 4.7.** *Let  $(w_{1n}, w_{2n})$  and  $M_q, N_q$  be defined as before. Then  $M_p = M_q, N_p = N_q$  for all  $p, q \in S$ .*

*Proof.* Step 1.  $\exists q \in S$ , there holds  $M_q N_q > 2(M_q + N_q)$ . Recall  $x_{1n}$  is the maximum point of  $u_{1n}$  and  $q_0 = \lim_{n \rightarrow \infty} \frac{x_{1n}}{|x_{1n}|} \in S$ ,  $r_n = 1/|x_{1n}|$ ,  $|q_0| = 1$ . Set

$$\bar{u}_{1n}(x) = u(x + x_{1n}), \bar{u}_{2n}(x) = u(x + x_{1n}).$$

By Lemma 4.3, one gets  $\bar{u}_{1n}(0) \geq -C$ . Also as  $|x_{1n}| \rightarrow \infty$ , by the standard elliptic estimates, we have  $\bar{u}_{1n} \rightarrow \bar{u}_1$  in  $C_{loc}^2(\mathbb{R}^2)$ . By Theorem 3.1, Harnack inequality and the arguments in Lemma 4.2, we shall have  $\bar{u}_{2n} \rightarrow \bar{u}_2$  in  $C_{loc}^2(\mathbb{R}^2)$  too. This yields that

$$\begin{cases} \Delta \bar{u}_1 + e^{\bar{u}_2} (1 - e^{\bar{u}_1}) = 0, \\ \Delta \bar{u}_2 + e^{\bar{u}_1} (1 - e^{\bar{u}_2}) = 0, \end{cases} \quad \text{in } \mathbb{R}^2$$

with  $\int_{\mathbb{R}^2} e^{\bar{u}_1} + e^{\bar{u}_2} dx < \infty$ . This means  $\bar{u}_1, \bar{u}_2$  are non-topological solutions. Denote

$$\int_{\mathbb{R}^2} e^{\bar{u}_2} (1 - e^{\bar{u}_1}) = 2\pi M_0, \int_{\mathbb{R}^2} e^{\bar{u}_1} (1 - e^{\bar{u}_2}) = 2\pi N_0.$$

By Lemma 4.1, we have

$$\int_{\mathbb{R}^2} e^{\bar{u}_1 + \bar{u}_2} dx = \pi(M_0 N_0 - 2M_0 - 2N_0) > 0.$$

By the Fatou's lemma, we have  $M_{q_0} \geq M_0, N_{q_0} \geq N_0$  and

$$\frac{1}{M_{q_0}} + \frac{1}{N_{q_0}} \leq \frac{1}{M_0} + \frac{1}{N_0} < \frac{1}{2}.$$

This ends the proof of the first step.

Step 2. If  $M_q N_q > 2(M_q + N_q)$ , we have  $w_{in}(q_{in}) = -\ln r_n^2 + O(1)$ . In fact, by (4.8), we have

$$\int_{B_d(q)} r_n^2 e^{w_{1n} + w_{2n}} dx = \pi(M_q N_q - 2M_q - 2N_q) + o(1) \geq c_0 > 0.$$

Thus

$$\max_{B_d(q)}(r_n^2 e^{w_{1n}}) \int_{B_d(q)} e^{w_{2n}} dx \geq \int_{B_d(q)} r_n^2 e^{w_{1n}+w_{2n}} dx \geq c_0.$$

This means that  $w_{1n}(q_{1n}) + 2 \ln r_n \geq -C$  uniformly. Also noting  $w_{1n} + 2 \ln r_n \leq 0$ , we have  $w_{kn}(q_{kn}) = -\ln r_n^2 + O(1)$ .

Step 3. For any  $p, q \in S$ ,

$$|w_{kn}(x) - w_{kn}(y)| \leq C, \quad \forall x \in \partial B_d(q), y \in \partial B_d(p), k = 1, 2.$$

By Green's representation formula, we have

$$|w_{1n}(x) - w_{1n}(y)| \leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \ln \frac{|x-z|}{|y-z|} e^{w_{2n}(z)} (1 - r_n^2 e^{w_{1n}(z)}) dz \right| + 2 \sum_{i=1}^{N_1} \left| \ln \frac{|x - r_n \epsilon_n p_{1i}|}{|y - r_n \epsilon_n p_{1i}|} \right|.$$

The uniform bound of the second term on the righthand side is obvious. We split the first term into two integral domain to estimate it. In  $|z-x| \leq \frac{d}{2}$  or  $|z-y| \leq \frac{d}{2}$ , we have  $e^{w_{2n}(z)}(1 - r_n^2 e^{w_{1n}(z)}) \rightarrow 0$  uniformly, therefore

$$\int_{\{|z-x| \leq \frac{d}{2}\} \cup \{|z-y| \leq \frac{d}{2}\}} \ln \frac{|x-z|}{|y-z|} e^{w_{2n}(z)} (1 - r_n^2 e^{w_{1n}(z)}) dz \rightarrow 0.$$

In  $\mathbb{R}^2 \setminus (\{|z-x| \leq \frac{d}{2}\} \cup \{|z-y| \leq \frac{d}{2}\})$ , we have if  $|z| \geq 2 \max(|x|, |y|)$ , then  $\frac{1}{3} \leq \frac{|x-z|}{|y-z|} \leq 3$ ; if  $|z| \leq 2 \max(|x|, |y|)$ , then  $\frac{d}{6|y|} \leq \frac{|x-z|}{|y-z|} \leq \frac{6|x|}{d}$ . Therefore, we have

$$\int_{\mathbb{R}^2 \setminus (\{|z-x| \leq \frac{d}{2}\} \cup \{|z-y| \leq \frac{d}{2}\})} \ln \frac{|x-z|}{|y-z|} e^{w_{2n}(z)} (1 - r_n^2 e^{w_{1n}(z)}) dz = O(1).$$

Step 4.  $M_p = M_q, N_p = N_q$  holds for all  $p, q \in S$ . Let  $q_0$  be the point as in Step 1, i.e.  $M_{q_0} N_{q_0} > 2(M_{q_0} + N_{q_0})$ . If  $M_p N_p > 2(M_p + N_p)$ , then we have by Step 2,  $w_{kn}(q_{0,kn}), w_{kn}(p_{kn}) = -2 \ln r_n + O(1), k = 1, 2$ . Picking up  $x \in \partial B_d(q), y \in \partial B_d(p)$ , we have by Lemma 4.6

$$w_{1n}(x) - w_{1n}(y) = (1 - \frac{M_{q_0,n}}{2})(-\ln r_n^2) - (1 - \frac{M_{p,n}}{2})(-\ln r_n^2) + O(1).$$

By Step 3, we must have  $M_p = M_q$ . Also we can get  $N_p = N_q$  by investigating  $w_{2n}(x) - w_{2n}(y)$ . If  $M_p N_p = 2(M_p + N_p)$ , then without loss of generality, we may assume  $M_p < M_{q_0}$ . By Lemma 4.6 and Step 1, we also have

$$w_{1n}(x) - w_{1n}(y) \leq C + (M_{q_0,n} - M_{p,n}) \ln r_n \rightarrow -\infty$$

contradicts to Step 3. This finishes the proof.  $\square$

By Lemma 4.7, we now denote  $M, N$  the mass at  $q \in S$  instead of  $M_q, N_q$  respectively. In the following two lemmas, we show the concentration may occur only for special values of  $\alpha_1, \alpha_2$ . The following lemma shows that there is no concentration at the origin.

**Lemma 4.8.** *For each constant  $0 < s < \min_{q \in S} |q|$ , we have*

$$\lim_{n \rightarrow \infty} \left[ \int_{|x| \leq s} e^{w_{1n}} (1 - r_n^2 e^{w_{2n}}) dx + \int_{|x| \leq s} e^{w_{2n}} (1 - r_n^2 e^{w_{1n}}) dx \right] = 0 \quad (4.13)$$

and  $w_{1n}, w_{2n} \rightarrow -\infty$  uniformly on  $B_s(0)$ . Moreover,  $|S| \geq 2, MN(|S| - 1) = 2NN_1 + 2MN_2$ .

*Proof.* Consider a small number  $s < \min_{q \in S} |q|$ . Suppose that

$$2\pi M_0 = \lim_{n \rightarrow \infty} \int_{|x| \leq s} e^{w_{2n}} (1 - r_n^2 e^{w_{1n}}) dx > 0, 2\pi N_0 = \lim_{n \rightarrow \infty} \int_{|x| \leq s} e^{w_{1n}} (1 - r_n^2 e^{w_{2n}}) dx.$$

Set

$$\tilde{w}_{in}(x) = w_{in}(x) - f_{i,r_n \epsilon_n}, f_{i,r_n \epsilon_n}(x) = 2 \sum_{j=1}^{N_i} \ln |x - r_n \epsilon_n p_{ij}|, i = 1, 2.$$

By previous discussion, we have

$$\tilde{w}_{in}(x) \rightarrow -\infty \text{ uniformly for any } K \subset \subset \mathbb{R}^2 \setminus (S \cup \{0\}).$$

By the Green's representation formula for  $\tilde{w}_{in}$  and the standard elliptic estimates, one can get

$$\begin{aligned} \tilde{w}_{1n}(x) - c_{1n} &\rightarrow -M_0 \ln |x| - \sum_{q \in S} M \ln |x - q|, \\ \tilde{w}_{2n}(x) - c_{2n} &\rightarrow -N_0 \ln |x| - \sum_{q \in S} N \ln |x - q|, \end{aligned}$$

in  $C_{loc}^2(\mathbb{R}^2 \setminus (S \cup \{0\}))$ . Recall the Pohozaev-type identity as in Lemma 4.1,

$$\begin{aligned} &\int_{\partial\Omega} (x \cdot \nu)(\nabla \tilde{w}_{1n} \cdot \nabla \tilde{w}_{2n}) dS - \int_{\partial\Omega} (x \cdot \nabla \tilde{w}_{1n})(\nu \cdot \nabla \tilde{w}_{2n}) dS - \int_{\partial\Omega} (x \cdot \nabla \tilde{w}_{2n})(\nabla \tilde{w}_{1n} \cdot \nu) dS \\ &= \int_{\partial\Omega} (x \cdot \nu)(e^{w_{1n}} + e^{w_{2n}} - r_n^2 e^{w_{1n}+w_{2n}}) dS - 2 \int_{\Omega} (e^{w_{1n}} + e^{w_{2n}} - r_n^2 e^{w_{1n}+w_{2n}}) dx \\ &\quad - \int_{\Omega} (x \cdot \nabla f_{1,r_n \epsilon_n})(e^{w_{1n}} - r_n^2 e^{w_{1n}+w_{2n}}) dx - \int_{\Omega} (x \cdot \nabla f_{2,r_n \epsilon_n})(e^{w_{2n}} - r_n^2 e^{w_{1n}+w_{2n}}) dx. \end{aligned}$$

Set

$$\begin{aligned} I_n &= \int_{\partial\Omega} (x \cdot \nu)(\nabla \tilde{w}_{1n} \cdot \nabla \tilde{w}_{2n}) dS - \int_{\partial\Omega} (x \cdot \nabla \tilde{w}_{1n})(\nu \cdot \nabla \tilde{w}_{2n}) dS - \int_{\partial\Omega} (x \cdot \nabla \tilde{w}_{2n})(\nabla \tilde{w}_{1n} \cdot \nu) dS \\ J_{1n} &= \int_{\Omega} (x \cdot \nabla f_{1,r_n \epsilon_n})(e^{w_{1n}} - r_n^2 e^{w_{1n}+w_{2n}}) dx, J_{2n} = \int_{\Omega} (x \cdot \nabla f_{2,r_n \epsilon_n})(e^{w_{2n}} - r_n^2 e^{w_{1n}+w_{2n}}) dx. \end{aligned}$$

First we take  $\Omega = \{|x| \leq s\}$ , then

$$I_n = -2\pi M_0 N_0 + o(1)$$

as  $n \rightarrow \infty$ . Moreover,

$$\begin{aligned} J_{1n} &= 4\pi N_1 N_0 + \sum_{i=1}^{N_1} \int_{|x| \leq s/r_n} \frac{2(x - \epsilon_n p_{1i}) \cdot \epsilon_n p_{1i}}{|x - \epsilon_n p_{1i}|^2} e^{u_{1n}} (1 - e^{u_{2n}}) dx + o(1) \\ &= 4\pi N_1 N_0 + o(1) \int_{|x| \leq R} \frac{2(x - \epsilon_n p_{1i}) \cdot \epsilon_n p_{1i}}{|x - \epsilon_n p_{1i}|^2} dx + R^{-1} \int_{|x| \geq R} e^{u_{1n}} (1 - e^{u_{2n}}) dx + o(1) \\ &= 4\pi N_1 N_0 + o(1) \end{aligned}$$

as  $n \rightarrow \infty$  and then  $R \rightarrow \infty$ . Also we have  $J_{2n} = 4\pi N_2 M_0 + o(1)$ . By the above estimates, we have by Pohozaev's identity,

$$-2\pi M_0 N_0 = -4\pi N_2 M_0 - 4\pi N_1 N_0 - 4\pi M_0 - 4\pi N_0 - 2 \int_{|x| \leq s} r_n^2 e^{w_{1n}+w_{2n}} dx + o(1).$$

Therefore we have

$$M_0 N_0 \geq 2(N_2 M_0 + N_1 N_0 + M_0 + N_0). \quad (4.14)$$

From (4.14), we get  $N_0 > 0$ . Next we take  $\Omega = \cup_{q \in S} B_r(q)$ ,  $r$  small enough. Note that

$$\begin{aligned} \nabla \tilde{w}_{1n} &\rightarrow \nabla \tilde{w}_1 = -M \frac{x-q}{|x-q|^2} + \nabla H_{1q}(x), \nabla \tilde{w}_{2n} \rightarrow \nabla \tilde{w}_2 = -N \frac{x-q}{|x-q|^2} + \nabla H_{2q}(x), \\ \nabla H_{1q}(x) &= -M_0 \frac{x}{|x|^2} - \sum_{p \in S \setminus \{q\}} M \frac{x-p}{|x-p|^2}, \nabla H_{2q}(x) = -N_0 \frac{x}{|x|^2} - \sum_{p \in S \setminus \{q\}} N \frac{x-p}{|x-p|^2}. \end{aligned}$$

As  $n \rightarrow \infty$ , we have

$$\begin{aligned} I_n &\rightarrow \int_{\partial\Omega} (x \cdot \nu)(\nabla \tilde{w}_1 \cdot \nabla \tilde{w}_2) dS - \int_{\partial\Omega} (x \cdot \nabla \tilde{w}_1)(\nu \cdot \nabla \tilde{w}_2) dS - \int_{\partial\Omega} (x \cdot \nabla \tilde{w}_2)(\nabla \tilde{w}_1 \cdot \nu) dS \\ &= -2\pi M N |S| + \sum_{q \in S} \int_{|x-q|=r} \frac{N}{r} q \cdot \nabla H_{1q} + \frac{M}{r} q \cdot \nabla H_{2q} dS + O(r) \\ &= -2\pi M N |S| - 2\pi N M_0 |S| - 2\pi M N_0 |S| - \sum_{q \in S} \sum_{p \neq q} 4\pi M N \frac{q \cdot (q-p)}{|q-p|^2} + O(r) \\ &= -2\pi |S| (M N |S| + N M_0 + M N_0) + O(r). \end{aligned}$$

From (4.8), we have

$$\begin{aligned} &2 \int_{\Omega} (e^{w_{1n}} + e^{w_{2n}} - r_n^2 e^{w_{1n}+w_{2n}}) dx \\ &= 4\pi |S| M + 4\pi |S| N + 2\pi |S| (M N - 2M - 2N) + o(1) \end{aligned}$$

Moreover, we have

$$\begin{aligned} J_{1n} &= 4\pi N_1 N |S| + \sum_{i=1}^{N_1} \int_{\Omega} \frac{2r_n \epsilon_n p_{1i} \cdot (x - r_n \epsilon_n p_{1i})}{|x - r_n \epsilon_n p_{1i}|^2} (e^{w_{1n}} - r_n^2 e^{w_{1n}+w_{2n}}) dx \\ &= 4\pi N N_1 |S| + o(1). \end{aligned}$$

Also we have  $J_{2n} = 4\pi M N_2 |S| + o(1)$ . Combining the above estimates, the Pohozaev's identity implies that

$$M N (|S| - 1) + M N_0 + N M_0 = 2N N_1 + 2M N_2. \quad (4.15)$$

By (4.14), one has

$$M_0 N_0 > 2N_2 M_0 + 2M_0 \Rightarrow N_0 > 2(N_2 + 1).$$

The same arguments also yield  $M_0 > 2(N_1 + 1)$ . Substituting  $M_0, N_0$  by these two inequalities we get

$$M N_0 + N M_0 > 2(N_2 + 1)M + 2(N_1 + 1)N$$

contradicts to (4.15). We must have  $M_0 = N_0 = 0$  which implies that

$$M N (|S| - 1) = 2N N_1 + 2M N_2.$$

This also implies  $|S| \geq 2$ .

The last thing we shall prove is  $w_{in} \rightarrow -\infty$  uniformly on  $B_s(0)$  for  $i = 1, 2$ . By Green's representation formula for  $u_{1n}$ , we have, for  $x \leq \frac{s}{2r_n}$ ,

$$u_{1n}(x) = f_{1n}(x) + C_{1n} + \frac{1}{2\pi} \int_{|y| \leq \frac{s}{r_n}} \ln \frac{|y|}{|x-y|} e^{u_{2n}} (1 - e^{u_{1n}}) dy + O(1).$$

Standard arguments([1]) shows that

$$u_{1n}(x) - f_{1n}(x) - C_{1n} = o(1) \ln r_n + O(1), \quad \text{for } |x| \leq \frac{s}{2r_n}.$$

Then it follows from Lemma 4.6 that  $C_{1n} = (M_{q,n} + 2N_1 + o(1)) \ln r_n + O(1)$ , and hence,  $w_{1n} = f_{1,r_n \epsilon_n}(x) + (M_{q,n} - 2 + o(1)) \ln r_n + O(1)$  for  $|x| \leq \frac{s}{2}$ . This ends the proof of present lemma.  $\square$

**Remark 4.1.** Even when all  $M_q$ 's and  $N_q$ 's are not the same, it is still true that there is no concentration at origin by repeating the computation in Lemma 4.8.

Next lemma shows that there is no concentration at  $\infty$ .

**Lemma 4.9.** For any  $R > \max_{q \in S} |q|$ ,  $w_{in} \rightarrow -\infty$  uniformly in  $|x| \geq R$  and

$$\lim_{n \rightarrow \infty} \int_{|x| \geq R} e^{w_{1n}} (1 - r_n^2 e^{w_{2n}}) dx + \int_{|x| \geq R} e^{w_{2n}} (1 - r_n^2 e^{w_{1n}}) dx = 0.$$

Moreover, we have  $MN(|S| + 1) = 2\beta_1 N + 2\beta_2 M$ .

*Proof.* Set

$$\varphi_{in}(x) = w_{in} \left( \frac{x}{|x|^2} \right) - 2\beta_i \ln |x|, i = 1, 2.$$

Then  $\varphi_{in}$  satisfies

$$\begin{cases} -\Delta \varphi_{1n} = |x|^{2\beta_2-4} e^{\varphi_{2n}} (1 - r_n^2 |x|^{2\beta_1} e^{\varphi_{1n}}), \\ -\Delta \varphi_{2n} = |x|^{2\beta_1-4} e^{\varphi_{1n}} (1 - r_n^2 |x|^{2\beta_2} e^{\varphi_{2n}}), \end{cases} \quad \text{in } |x| < R$$

for  $R < (\max_{q \in S} |q|)^{-1}$ . Denote  $\tilde{S} = \{\frac{q}{|q|^2} | q \in S\}$ . Then by previous arguments, we have  $\varphi_{in} \rightarrow -\infty, i = 1, 2$  uniformly for any compact set  $K \subset \mathbb{R}^2 \setminus (\tilde{S} \cup \{0\})$ .

For  $0 < s < \frac{1}{4}(\max_{q \in S} |q|)^{-1}$ , set

$$\begin{aligned} \lim_{n \rightarrow \infty} \inf \int_{|x| \leq s} |x|^{2\beta_2-4} e^{\varphi_{2n}} (1 - r_n^2 |x|^{2\beta_1} e^{\varphi_{1n}}) dx &= 2\pi L_1, \\ \lim_{n \rightarrow \infty} \inf \int_{|x| \leq s} |x|^{2\beta_1-4} e^{\varphi_{1n}} (1 - r_n^2 |x|^{2\beta_2} e^{\varphi_{2n}}) dx &= 2\pi L_2. \end{aligned}$$

We need to show  $\varphi_{in} \rightarrow -\infty$  uniformly in  $B_s$ . If not, we may assume  $\max_{|x| \leq s} \varphi_{2n} \geq -C$ . Then we must have  $\max_{|x| \leq s} (\varphi_{1n}, \varphi_{2n}) \rightarrow +\infty$ . Otherwise, as  $\beta_1 > 1$ , we will have by Harnack inequality that  $\varphi_{2n}$  is uniformly bounded in  $B_s$  which contradicts to  $\varphi_{2n} \rightarrow -\infty$  on  $\partial B_s$ . Denote  $\lambda_{in} = \varphi_{in}(y_{in}) = \max_{|x| \leq s} \varphi_{in}$ . Set

$$t_n = \min\{e^{-\frac{\lambda_{1n}}{2\beta_1-2}}, e^{-\frac{\lambda_{2n}}{2\beta_2-2}}\} = e^{-\frac{\lambda_{1n}}{2\beta_1-2}} \rightarrow 0.$$

Then we have  $y_{1n} \rightarrow 0$  and  $\lambda_{1n} \rightarrow +\infty$ . Next, we shall discuss in two cases:

Case 1.  $\frac{|y_{1n}|}{t_n} \leq C$ . Set

$$\bar{\varphi}_{in}(x) = \varphi_{in}(t_n x) - \lambda_{in}, i = 1, 2.$$

Then we have  $\bar{\varphi}_{in} \leq 0, i = 1, 2$ . and

$$\begin{cases} -\Delta \bar{\varphi}_{1n} = e^{\lambda_{2n} + 2(\beta_2-1) \ln t_n} |x|^{2\beta_2-4} e^{\bar{\varphi}_{2n}} (1 - r_n^2 t_n^2 |x|^{2\beta_1} e^{\bar{\varphi}_{1n}}), \\ -\Delta \bar{\varphi}_{2n} = |x|^{2\beta_1-4} e^{\bar{\varphi}_{1n}} (1 - r_n^2 t_n^{2\beta_2} e^{\lambda_{2n}} |x|^{2\beta_2} e^{\bar{\varphi}_{2n}}), \end{cases} \quad \text{in } |x| \leq s/t_n.$$

Also we have  $\bar{\varphi}_{1n}(x) \leq \bar{\varphi}_{1n}(\frac{y_{1n}}{t_n}) = 0$ . Then by standard elliptic estimates and  $\lambda_{2n} + 2(\beta_2 - 1) \ln t_n \leq 0$ , we have  $\bar{\varphi}_{1n} \rightarrow \varphi_1$  in  $W_{loc}^{2,\gamma}(\mathbb{R}^2)$  for some  $\gamma > 1$ . As for  $\bar{\varphi}_{2n}$ , by Harnack inequality, one gets either  $\bar{\varphi}_{2n} \rightarrow -\infty$  locally uniformly in  $\mathbb{R}^2$  or  $\bar{\varphi}_{2n} \rightarrow \bar{\varphi}_2$  in  $W_{loc}^{2,\gamma}(\mathbb{R}^2)$ . Following the same arguments as in Lemma 4.2, we can exclude the previous case. In fact, by the same arguments, we also have  $\lambda_{2n} + 2(\beta_2 - 1) \ln t_n \geq -C$ , otherwise,

$$\Delta \bar{\varphi}_1 = 0, \bar{\varphi}_1(x_0) = 0, x_0 = \lim_{n \rightarrow \infty} \frac{y_{1n}}{t_n} \Rightarrow \bar{\varphi}_1 \equiv 0$$

which contradicts to  $\int_{\mathbb{R}^2} |x|^{2\beta_1-4} e^{\bar{\varphi}_1} dx < +\infty$  by Fatou's lemma. And  $\bar{\varphi}_1, \bar{\varphi}_2$  satisfy

$$\begin{cases} -\Delta \bar{\varphi}_1 = c_0 |x|^{2\beta_2-4} e^{\bar{\varphi}_2}, \\ -\Delta \bar{\varphi}_2 = |x|^{2\beta_1-4} e^{\bar{\varphi}_1}, \end{cases} \text{ in } \mathbb{R}^2$$

with  $|x|^{2\beta_1-4} e^{\bar{\varphi}_1}, |x|^{2\beta_2-4} e^{\bar{\varphi}_2} \in L^1(\mathbb{R}^2)$  for some constant  $0 < c_0 \leq 1$ . Set

$$A_1 = \frac{1}{2\pi} \int_{\mathbb{R}^2} c_0 |x|^{2\beta_2-4} e^{\bar{\varphi}_2} dx, A_2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2\beta_1-4} e^{\bar{\varphi}_1} dx.$$

By Pohozaev's identity and repeating the arguments in Lemma 4.1, one gets

$$A_1 A_2 = 2(\beta_2 - 1) A_1 + 2(\beta_1 - 1) A_2.$$

Noting  $L_1 \geq A_1, L_2 \geq A_2$ , we have

$$L_1 L_2 \geq 2(\beta_2 - 1) L_1 + 2(\beta_1 - 1) L_2. \quad (4.16)$$

Case 2.  $\frac{|y_{1n}|}{t_n} \rightarrow +\infty$ . Recall the definition of  $w_{in}, i = 1, 2$  in (4.6) and set

$$\bar{w}_{in}(x) = w_{in}(x/|y_{1n}|) - 2 \ln |y_{1n}|, i = 1, 2.$$

$\bar{w}_{in}$  should satisfy

$$\begin{cases} \Delta \bar{w}_{1n} + e^{\bar{w}_{2n}} (1 - r_n^2 |y_{1n}|^2 e^{\bar{w}_{1n}}) = 4\pi \sum_{i=1}^{N_1} \delta_{r_n \epsilon_n |y_{1n}| p_{1i}}, \\ \Delta \bar{w}_{2n} + e^{\bar{w}_{1n}} (1 - r_n^2 |y_{1n}|^2 e^{\bar{w}_{2n}}) = 4\pi \sum_{i=1}^{N_2} \delta_{r_n \epsilon_n |y_{1n}| p_{2i}}, \end{cases} \text{ in } \mathbb{R}^2.$$

Notice that

$$\bar{w}_{1n} \left( \frac{y_{1n}}{|y_{1n}|} \right) = \varphi_{1n}(y_{1n}) + 2(\beta_1 - 1) \ln |y_{1n}| = 2(\beta_1 - 1) \ln \frac{|y_{1n}|}{t_n} \rightarrow +\infty.$$

By Theorem 3.1, we know that  $\bar{w}_{2n}(x)$  also blows up at  $q^* = \lim_{n \rightarrow \infty} \frac{y_{1n}}{|y_{1n}|} \in \mathbb{S}^1$ . This implies that along a subsequence,  $\bar{w}_{in}$  has a finite non-empty blow up set  $S^*$  and the elements in  $S^*$  are non-zero points.

Turning back to  $\varphi_{in}, i = 1, 2$  and following the same arguments as in Lemma 4.8, we may assume

$$\begin{aligned} |x|^{2\beta_2-4} e^{\varphi_{2n}} (1 - r_n^2 |x|^{2\beta_1} e^{\varphi_{1n}}) &\rightarrow 2\pi L_1 \delta_0 + 2\pi M \sum_{q \in \tilde{S}} \delta_q, \\ |x|^{2\beta_1-4} e^{\varphi_{1n}} (1 - r_n^2 |x|^{2\beta_2} e^{\varphi_{2n}}) &\rightarrow 2\pi L_2 \delta_0 + 2\pi N \sum_{q \in \tilde{S}} \delta_q \end{aligned}$$

and also

$$\varphi_{1n}(x) - c_{1n} \rightarrow -L_1 \ln |x| - M \sum_{q \in \tilde{S}} \ln |x - q|, \varphi_{2n}(x) - c_{2n} \rightarrow -L_2 \ln |x| - N \sum_{q \in \tilde{S}} \ln |x - q|$$

in  $C_{loc}^1(\mathbb{R}^2 \setminus (\tilde{S} \cup \{0\}))$  for some constants  $c_{1n}, c_{2n} \rightarrow -\infty$ . Also we have the following Pohozaev's identity

$$\begin{aligned} & \int_{\partial\Omega} (x \cdot \nabla \varphi_{2n})(\nu \cdot \nabla \varphi_{1n}) dS + \int_{\partial\Omega} (x \cdot \nabla \varphi_{1n})(\nu \cdot \nabla \varphi_{2n}) dS - \int_{\partial\Omega} (x \cdot \nu)(\nabla \varphi_{2n} \cdot \nabla \varphi_{1n}) dS + o(1) \\ &= 2(\beta_2 - 1) \int_{\Omega} |x|^{2\beta_2 - 4} e^{\varphi_{2n}} dx + 2(\beta_1 - 1) \int_{\Omega} |x|^{2\beta_1 - 4} e^{\varphi_{1n}} dx - 2(\beta_1 + \beta_2 - 1) \times \\ & \int_{\Omega} r_n^2 |x|^{2\beta_1 + 2\beta_2 - 4} e^{\varphi_{1n} + \varphi_{2n}} dx. \end{aligned}$$

Repeating the arguments of Lemma 4.8 in  $\Omega = \cup_{q \in \tilde{S}} B_r(q)$ , we obtain that

$$MN(|S| + 1) + NL_1 + ML_2 = 2\beta_1 N + 2\beta_2 M. \quad (4.17)$$

If Case 1 happens, by (4.16), we have  $L_i \geq 2(\beta_i - 1)$  which implies

$$MN(|S| + 1) + NL_1 + ML_2 \geq 2\beta_1 N + 2\beta_2 M + (MN - 2M - 2N).$$

This yields a contradiction to (4.17) as  $MN - 2M - 2N > 0$ . Hence, we must have Case 2 happens. Then by the proof of Lemma 4.8, we have

$$o(1) = \int_{|x| \leq d} e^{\bar{w}_{2n}} (1 - r_n^2 |y_{1n}|^2 e^{\bar{w}_{1n}}) dx = \int_{|x| \leq d/|y_{1n}|} e^{w_{2n}} (1 - r_n^2 e^{w_{1n}}) dx \geq 2\pi |S| M$$

for some small  $d > 0$  which yields a contradiction. This implies that  $L_1 = L_2 = 0$

$$MN(|S| + 1) = 2\beta_1 N + 2\beta_2 M \quad (4.18)$$

and  $\varphi_{in} \rightarrow -\infty$  uniformly in  $B_s(0)$ .  $\square$

Now Lemma 4.8 and Lemma 4.9 tell us that there is no concentration of mass at 0 and  $\infty$  for  $w_{in}, i = 1, 2$ . This implies that

$$|S|M = 2(\beta_1 + N_1), |S|N = 2(\beta_2 + N_2). \quad (4.19)$$

Combining (4.15), (4.18) and (4.19) together, we can get

$$|S| = \frac{(\beta_1 + N_1)(\beta_2 + N_2)}{\beta_1 \beta_2 - N_1 N_2}. \quad (4.20)$$

This implies that

$$\frac{N_1}{\beta_1 + N_1} + \frac{N_2}{\beta_2 + N_2} = \frac{|S| - 1}{|S|} \in \left\{ \frac{k-1}{k}, k = 2, \dots, \max(N_1, N_2) \right\}. \quad (4.21)$$

From (4.3), we know that  $\frac{N_1+1}{\beta_1+N_1} + \frac{N_2+1}{\beta_2+N_2} < 1$ . Combining with (4.21), we get  $\frac{1}{\beta_1+N_1} + \frac{1}{\beta_2+N_2} < \frac{1}{k}$ . Then we have

$$\frac{k-1}{k} \leq \frac{\max(N_1, N_2)}{\beta_1 + N_1} + \frac{\max(N_1, N_2)}{\beta_2 + N_2} < \frac{\max(N_1, N_2)}{k}$$

which implies  $k \leq \max(N_1, N_2)$ . Theorem 4.1 follows from (4.21) immediately.



## 5 The existence of non-topological solutions

Set  $v_{i\epsilon} = u_{i\epsilon} - h_{i\epsilon}$ , where  $h_{i\epsilon} = 2 \sum_{j=1}^{N_i} \ln |x - \epsilon p_{ij}| - (N_i + \beta_i) \ln(1 + |x|^2)$ ,  $i = 1, 2$ . Then  $v_{i\epsilon}$  satisfies that

$$\begin{aligned} \Delta v_{1\epsilon} + e^{v_{2\epsilon} + h_{2\epsilon}} (1 - e^{v_{1\epsilon} + h_{1\epsilon}}) &= \frac{4(N_1 + \beta_1)}{(1 + |x|^2)^2} = g_1, \\ \Delta v_{2\epsilon} + e^{v_{1\epsilon} + h_{1\epsilon}} (1 - e^{v_{2\epsilon} + h_{2\epsilon}}) &= \frac{4(N_2 + \beta_2)}{(1 + |x|^2)^2} = g_2. \end{aligned} \quad (5.1)$$

**Theorem 5.1.** *Under the assumption of Theorem 4.1, we have*

$$|v_{1\epsilon}|_{L^\infty(\mathbb{R}^2)} + |v_{2\epsilon}|_{L^\infty(\mathbb{R}^2)} \leq C$$

for some constant  $C$  independent of  $\epsilon$ .

*Proof.* By Theorem 4.1, we have  $v_{i\epsilon}$  is bounded in  $L_{loc}^\infty(\mathbb{R}^2)$ . Consider the function

$$\xi_{i\epsilon}(x) = u_{i\epsilon} \left( \frac{x}{|x|^2} \right) - 2\beta_i \ln |x|, \text{ for } |x| \leq R_0 := (1 + \max_{i,j} |p_{ij}|)^{-1}.$$

Obviously,  $\xi_{i\epsilon}(x) \in L_{loc}^\infty(B_{R_0} \setminus \{0\})$  and

$$\begin{aligned} \Delta \xi_{1\epsilon} + |x|^{2\beta_2 - 4} e^{\xi_{2\epsilon}} (1 - |x|^{2\beta_1} e^{\xi_{1\epsilon}}) &= 0, \\ \Delta \xi_{2\epsilon} + |x|^{2\beta_1 - 4} e^{\xi_{1\epsilon}} (1 - |x|^{2\beta_2} e^{\xi_{2\epsilon}}) &= 0, \end{aligned} \quad \text{in } |x| \leq R_0. \quad (5.2)$$

We now show  $\xi_{i\epsilon}$ ,  $i = 1, 2$  is bounded from above in  $B_{R_0}(0)$ . If not, we may assume a sequence of  $\lambda_{1n} = \xi_{1n}(y_{1n}) = \max_{|x| \leq R_0} \xi_{1n} \rightarrow +\infty$ . In fact, we must have  $\lambda_{2n} = \xi_{2n}(y_{2n}) = \max_{|x| \leq R_0} \xi_{2n} \rightarrow +\infty$ . Otherwise, by Harnack inequality, we will have  $\xi_{1n} \rightarrow +\infty$  in  $B_{R_0}$  which contradicts to  $\xi_{1n}(x) \in L^\infty(\partial B_{R_0})$ .

We claim that

$$\max(\xi_{1n}(y_{1n}) + 2(\beta_1 - 1) \ln |y_{1n}|, \xi_{2n}(y_{2n}) + 2(\beta_2 - 1) \ln |y_{2n}|) \rightarrow +\infty.$$

Otherwise, without loss of generality, we can choose  $t_n$  such that

$$\xi_{1n}(y_{1n}) + 2(\beta_1 - 1) \ln t_n = 0, \xi_{2n}(y_{2n}) + 2(\beta_2 - 1) \ln t_n \leq 0.$$

Consider the scaled function  $\zeta_{in}(x) = \xi_{in}(t_n x) - \xi_{in}(y_{in})$ . Set

$$L_1 = \lim_{n \rightarrow \infty} \int_{|x| \leq R_0} |x|^{2\beta_2 - 4} e^{\xi_{2n}} (1 - |x|^{2\beta_1} e^{\xi_{1n}}) dx, L_2 = \lim_{n \rightarrow \infty} \int_{|x| \leq R_0} |x|^{2\beta_1 - 4} e^{\xi_{1n}} (1 - |x|^{2\beta_2} e^{\xi_{2n}}) dx.$$

Repeating the same arguments as in Lemma 4.9, we must have

$$\frac{2(\beta_1 - 1)}{L_1} + \frac{2(\beta_2 - 1)}{L_2} \leq 1.$$

As  $L_1 \leq 2(\beta_1 + N_1)$ ,  $L_2 \leq 2(\beta_2 + N_2)$ , substituting this to the above inequality yields that

$$(\beta_1 - 1)(\beta_2 - 1) \leq (N_1 + 1)(N_2 + 1)$$

which is a contradiction to (4.3). This proves the claim.

We may assume  $\xi_{1n}(y_{1n}) + 2(\beta_1 - 1) \ln |y_{1n}| \rightarrow +\infty$ . Set  $\bar{u}_{in}(x) = u_{in}(x/|y_{1n}|) - 2 \ln |y_{1n}|$ . Noting  $\bar{u}_{1n}(y_{1n}/|y_{1n}|) \rightarrow +\infty$  and the equations  $\bar{u}_{in}$  satisfying, along a subsequence,  $\bar{u}_{in}$  satisfies

the blow up situation in Theorem 3.1 and the blow up set  $\bar{S}$  contains at least one non-zero point on  $\mathbb{S}^1$ . Then the proof of Lemma 4.8 implies that

$$o(1) = \int_{|x| \leq d} e^{\bar{u}_{1n}} (1 - |y_{1n}|^2 e^{\bar{u}_{2n}}) dx = \int_{|y| \leq d/|y_{1n}|} e^{u_{1n}} (1 - e^{u_{2n}}) dx.$$

Theorem 4.1 tells us that  $u_{in}$  is uniformly bounded in  $C_{loc}^2(\mathbb{R}^2 \setminus B_R(0))$ ,  $R > \max(|p_{ij}| + 1)$ . From  $\int_{\mathbb{R}^2} e^{u_{2n}} dx \leq C < \infty$ , one gets  $|\{x | u_{2n}(x) > -1\}| \leq C_1$ . Now we have

$$\int_{R \leq |y| \leq C_2 R} e^{u_{1n}} (1 - e^{u_{2n}}) dx \geq c(1 - e^{-1})(C_2 R^2 - C_1) \geq C_0 > 0$$

for some fixed  $C_2$  large enough. This yields a contradiction.

By now, we have proved that  $\xi_i$  is bounded from above. Then if we note that  $\xi_i \in L^\infty(\partial B_{R_0})$ , by standard elliptic estimates, we can prove  $\xi_i$  is uniformly bounded from below too.  $\square$

Due to Theorem 5.1, we now can calculate Leray-Schauder degree for (4.1). Recall the following Hilbert space defined in Section 2 for  $\beta = \min(\beta_1, \beta_2, 2) > 1$

$$\mathcal{D} = \{v : \mathbb{R}^2 \rightarrow \mathbb{R} \mid |v|_{\mathcal{D}}^2 = \int_{\mathbb{R}^2} |\nabla v|^2 dx + \int_{\mathbb{R}^2} \frac{v^2}{(1 + |x|^2)^\beta} dx < +\infty\}.$$

For every  $v \in \mathcal{D}$ , there holds that for any  $\beta > 1$

$$\ln \int_{\mathbb{R}^2} \frac{e^v}{(1 + |x|^2)^\beta} dx \leq \frac{1}{8\pi\gamma} |\nabla v|_{L^2}^2 + \bar{v} + C_\gamma \quad (5.3)$$

for any positive  $\gamma < 2(\beta - 1)$  if  $1 < \beta < 2$  and  $\gamma = 2$  if  $\beta \geq 2$ . Here,

$$\bar{v} = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{v}{(1 + |x|^2)^\beta} dx.$$

The above inequality can be found in [14].

**Lemma 5.1.** *Under the assumption of Theorem 4.1, we have*

$$|v_{1\epsilon}|_{\mathcal{D}} + |v_{2\epsilon}|_{\mathcal{D}} \leq C$$

for some constant  $C$  independent of  $\epsilon$ .

The proof of Lemma 5.1 is simply integrating by parts and using Theorem 5.1. We omit the details here.

**The proof for Theorem 1.2:** Now we can prove Theorem 1.2 by Leray-Schauder degree theory. We now define a map as follows.

$$T(\epsilon, v_1, v_2) = (T_1(\epsilon, v_1, v_2), T_2(\epsilon, v_1, v_2)) : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D} \times \mathcal{D}$$

for  $\epsilon \in [0, 1]$  by

$$\begin{aligned} T_1(\epsilon, v_1, v_2) &= (-\Delta + \sigma)^{-1} [e^{v_2 + h_{2\epsilon}} (1 - e^{v_1 + h_{1\epsilon}}) + \sigma v_1 - g_1], \\ T_2(\epsilon, v_1, v_2) &= (-\Delta + \sigma)^{-1} [e^{v_1 + h_{1\epsilon}} (1 - e^{v_2 + h_{2\epsilon}}) + \sigma v_2 - g_2] \end{aligned} \quad (5.4)$$

where  $\sigma = \frac{1}{(1 + |x|^2)^\beta}$ .

It is obvious that  $T(\epsilon, v_1, v_2) : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D} \times \mathcal{D}$  is compact by (5.3). And by Lemma 5.1, there exists a constant  $R > 0$  such that every zero of  $I - T(\epsilon, v_1, v_2)$  is contained in a ball  $\Omega_R = \{(v_1, v_2) \in \mathcal{D} \times \mathcal{D} \mid |v_1|_{\mathcal{D}} + |v_2|_{\mathcal{D}} < R\}$ . Then the degree  $\deg(I - T(\epsilon, v_1, v_2), \Omega_R, 0)$  is well defined.

Moreover,  $I - T(\epsilon, v_1, v_2)$  is a continuous homotopy with respect to  $\epsilon$  and preserves degree by Lemma 5.1.

By the above arguments, we only need to calculate the Leray-Schauder degree of  $I - T(0, v_1, v_2)$ . It is well known that non-radial solutions of  $I - T(0, v_1, v_2) = 0$ , if they exist, do not affect the calculation of  $\deg(I - T(0, v_1, v_2), \Omega_R, 0)$ . See [24] and references therein. Then by the proof of Theorem 1.1, we have

$$\deg(I - T(1, v_1, v_2), \Omega_R, 0) = \deg(I - T(0, v_1, v_2), \Omega_R, 0) = -1.$$

This proves Theorem 1.2.

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